Topics in Applied Mathematics

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Fourier Series and Fourier Integrals

In natural and engineering sciences we often deal with periodic processes such as oscillatory or rotary motion of various parts of machines and apparatuses, periodic motion of heavenly bodies and elementary particles, acoustics and electromagnetic vibrations etc.

These processes are describes mathematically by means of ***periodic functions***.

1. Fourier Series

Definition: A function $f(t)$ of one independent variable t is said to be **periodic** if there is a number $T > 0$ such that $f(t+T) = f(t) \ \ \forall t \in D_f$.

The number T (which is **the least** of such values) is called a **period** of f.

Examples:

(1) Each of the functions $f_1(t) = \sin t$ and $f_2(t) = \cos t$ are periodic with period $T = 2\pi$. There is no positive value T; $0 < T < 2\pi \ \ni \sin(t+T) = \sin t \ \ \forall t\,, -\infty < t < \infty$; or $\cos(t+T) = \cos t \ \ \forall t$. In fact, each of these functions has period $2\pi, 4\pi, 6\pi, \ldots$

(2) The function $f(t) = \tan t$ is undefined for values of t given by $t = \frac{\pi}{2} + m\pi\,,\ m \in \mathbb{Z}$. Neverthless, the function is periodic because, for every $t \in D_f,\ tan(t+m\pi) = \tan t$. Of course, m must be positive for $m\pi$ to be a period. The ***fundamental period*** of $\tan t$ is π.

(3) The function $f(t) = \sin 3t$ is periodic with period $\frac{2\pi}{3}$. In general, if ω is a nonzero positive constant, the function $f(t) = \sin \omega t$ is periodic with period $\frac{2\pi}{\omega}$.

(4) Let f be a constant function defined on $T \le t < \infty$ by $f(t) = c$. For every $T > 0$ and for every $t \in D_f$; f$(t+T) = f(t) = c$. Hence, f is ***periodic*** (by definition). Since there is no smallest positive number, f ***has no fundamental period***. That is, a constant quantity is a ***periodic*** function ***with an arbitrary period*** (i.e. every number is its period).

Remark 1: It is easy to verify that if T is a period of f, and m a positive integer, then mT is also a period of f. We call T the "***fundamental period***" or simply the "***period***" if it is the smallest period of f. Note that **example 4** above shows that some periodic functions have no ***fundamental period***.

Remark 2: If f and g are two ***periodic*** function with ***common period*** T, then their sum $f+g$ is also ***periodic*** with period T. To prove this,
Let $F(t)=f(t)+g(t)$. Then $F(t+T)=f(t+T)+g(t+T)=f(t)+g(t)=F(t)$. Thus, F which is the sum of f and g is a periodic function.

Theorem: The functions $\sin\left(\frac{m\pi t}{L}\right)$ and $\cos\left(\frac{m\pi t}{L}\right)$, $m=1,2,3,\ldots$, are periodic with ***fundamental period*** $T=\frac{2L}{m}$. Further, every such functions has the period $2L$.
Proof: We must show that there are values of T for which

$$\sin\frac{m\pi(t+T)}{L}=\sin\frac{m\pi t}{L}\quad \forall t \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and that the minimum such value of T is $\frac{2L}{m}$.
Expanding the left side of Eq(1) gives

$$\sin\frac{m\pi(t+T)}{L}=\sin\frac{m\pi t}{L}\cos\frac{m\pi T}{L}+cos\frac{m\pi t}{L}\sin\frac{m\pi T}{L}=\sin\frac{m\pi t}{L}\ldots\ldots\ldots\ldots (2)$$

Eq(2) is satisfied $\forall t$ if we can choose T so that $\cos\frac{m\pi T}{L}=1$ and $\sin\frac{m\pi T}{L}=0$. These requirements are met if we take $\frac{m\pi T}{L}=0,\pm 2\pi,\pm 4\pi,\ldots$

Now, to show that there are no other values of T for which Eq(1) is satisfied. Assume that there is such a , and choose t so that $cos\frac{m\pi t}{L}=0$. Then Eq(2) becomes

$$\sin\frac{m\pi t}{L}\cos\frac{m\pi T}{L}=\sin\frac{m\pi t}{L}$$

which is true only for the values of $\frac{m\pi T}{L}$ given above.

The fundamental period of $\sin\frac{m\pi t}{L}$ is then

$$T = \frac{2\pi}{\frac{m\pi}{L}} = \frac{2L}{m} \quad \dots\dots\dots\dots\dots\dots (3)$$

Since any interval multiple of a period is also a period, it follows that $\sin\frac{m\pi t}{L}$ has the period $2L$ for any m. A similar argument applies to $\cos\frac{m\pi t}{L}$.

Orthogonality of Trigonometric Functions

Definition: The standard ***inner product*** (u, v) of two real-valued functions u and v on the interval $a \le t \le b$ is defined by

$$(u, v) = \int_a^b u(t)v(t)dt \quad \dots\dots\dots\dots\dots (4)$$

The functions u and v are said to be **orthogonal** on $[a, b]$ if their inner product vanishes, i.e.

$$\int_a^b u(t)v(t)dt = 0 \quad \dots\dots\dots\dots (5)$$

Definition: A set of functions is said to be "**musually orthogonal**" if each distinct pair of functions in the set is orthogonal.

Example: The orthogonal trigonometric system $\{1, cosx, cos2x, \dots, sinx, sin2x, \dots\}$ on $[-\pi, \pi]$ has the orthogonal property. Note that,

$$\sin\alpha \sin\beta = \frac{1}{2}\big(cos(\alpha - \beta) - cos(\alpha + \beta)\big)$$

and

$$\cos\alpha \cos\beta = \frac{1}{2}\big(cos(\alpha + \beta) + cos(\alpha - \beta)\big)$$

The following theorem relates these concepts to the functions $\sin\frac{m\pi t}{L}$ and $\cos\frac{m\pi t}{L}$.

Theorem: The functions 1, $\sin\frac{m\pi t}{L}$, and $\cos\frac{m\pi t}{L}$, $m = 1,2,3, \dots$ form a mutually orthogonal set of functions on $[-L, L]$ as well as on every interval $[a, a + 2L]$. In fact, they satisfy the following equations, known as orthogonality relations:

$$(1)\int_{-L}^{L}\cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt=\begin{cases}0 & if\ m\neq n\\ L & if\ m=n\end{cases}\ \ldots\ldots\ldots\ldots\ (6)$$

$$(2)\int_{-L}^{L}\cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$(3)\int_{-L}^{L}\sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt=\begin{cases}0 & if\ m\neq n\\ L & if\ m=n\end{cases}\ \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$(4)\int_{-L}^{L}\frac{1}{2}\sin\frac{m\pi t}{L}\,dt=-\frac{1}{2m\pi}cos\frac{m\pi t}{L}\Big|_{-L}^{L}=-\frac{1}{2m\pi}(-1)^m-(-1)^m=0\ \ldots\ldots\ldots(9)$$

$$(5)\int_{-L}^{L}\frac{1}{2}\cos\frac{m\pi t}{L}\,dt=\frac{1}{2m\pi}sin\frac{m\pi t}{L}\Big|_{-L}^{L}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$(6)\int_{-L}^{L}\cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt=\frac{1}{2}\int_{-L}^{L}\cos\frac{(m+n)\pi t}{L}dt+\frac{1}{2}\int_{-L}^{L}\cos\frac{(m-n)\pi t}{L}dt=0\ldots.(11)$$

The results in Eqs.(6-8) can be obtained by direct integration. Let us derive, for example, Eq(8). Using the equality

$$\sin\alpha\sin\beta=\frac{1}{2}\big(cos(\alpha-\beta)-cos(\alpha+\beta)\big)$$

We have for $m\neq n$ that

$$\int_{-L}^{L}\sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt=\frac{1}{2}\int_{-L}^{L}\cos\frac{(m-n)\pi t}{L}dt-\frac{1}{2}\int_{-L}^{L}\cos\frac{(m+n)\pi t}{L}dt.$$

$$=\frac{1}{2}\left(\frac{\sin\frac{(m-n)\pi t}{L}}{\frac{(m-n)\pi}{L}}\right)\Bigg|_{-L}^{L}-\frac{1}{2}\left(\frac{\sin\frac{(m+n)\pi t}{L}}{\frac{(m+n)\pi}{L}}\right)\Bigg|_{-L}^{L}=0$$ as long as $m+n$ and $m-n$ are not zero.

Since m and n are positive, then $m+n\neq 0$.

On the other hand, if $m-n=0\rightarrow n=n$, and the integrals must be evaluated in a different way. In this case,

$$\int_{-L}^{L}\sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt=\int_{-L}^{L}\left(\sin\frac{m\pi t}{L}\right)^2dt=\frac{1}{2}\int_{-L}^{L}-1\,dt-\frac{1}{2}\int_{-L}^{L}\cos\frac{2m\pi t}{L}dt=L$$

The other identities can be proved in a similar manner.

Definition: If $\varphi(t)$ is a function on $[a,b]$, we define the norm of φ to be,

$$\|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}} = \left[\int_a^b (\varphi(t))^2\ dt\right]^{\frac{1}{2}}$$

Definition: A collection of functions $\{\varphi_0(t), \varphi_1(t), \dots, \varphi_m(t), \dots\}$ defined on $[a,b]$ is called **orthonormal** on $[a,b]$ if,

$$\big(\varphi_i(t), \varphi_j(t)\big) = \int_a^b \varphi_i(t)\ \varphi_j(t)\ dt = \begin{cases} 0 & if\ i \neq j \\ l & if\ i = j \end{cases}$$

Remark: If the collection $\{\varphi_0(t), \varphi_1(t), \dots, \varphi_m(t), \dots\}$ is orthogonal on $[a,b]$ and $\|\varphi_i\| \neq 0$ where $\|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}} = \left[\int_a^b (\varphi(t))^2\ dt\right]^{\frac{1}{2}}$,the collection $\left\{\frac{\varphi_0(t)}{\|\varphi_0\|}, \frac{\varphi_1(t)}{\|\varphi_1\|}, \dots, \frac{\varphi_m(t)}{\|\varphi_m\|}, \dots\right\}$ is **orthonormal** on $[a,b]$.

1.1 Formulation of Fourier Series

From the above discussion, it is natural to ask whether constants a_0, a_n, and b_n, $n = 1,2, \dots$ can be found such that a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \dots\dots\dots\dots\dots\dots\dots (A)$$

will represent an arbitrary function $f(x)$ of period $2L$ at each point of its domain.

That is, the main aim of the present subsection is to elucidate the following questions:

(1) What are the periodic functions with period $2L$ which can be expanded in trigonometric series (A) ?
(2) How can we determine the coefficients a_0, a_n, and b_n, $n = 1,2, \dots$ of expansion (A) if it is valid ?
(3) What is the connection between the character of convergence of series (A) and the properties of the function (x) ?

We shall begin to consider the series in the right-hand side of Eq(A).

(a) On the set of points where the series converges, it defines a function f, whose value at each point is the sum of the series for that value of *x*. In this case the series (A) is said to be the **Fourier series**.

(b) The series is also called the trigonometric series.

(c) A relation of form (A) (provided it is valid) is called **the expansion** of the function $f(x)$ into a trigonometric series (or Fourier series).

Note that the first term in the series of Eq(A) is written as $\frac{a_0}{2}$ rather than as a_0 in order to simplify a formula for the coefficients that we will derive bellow. That is, the introduction of the factor $\frac{1}{2}$ is a conventional device to render more symmetric the final formulas for the coefficients.

Now let us discuss the problem of **determining the coefficients** a_0, a_n, and b_n, $n = 1,2, \ldots$ of expansion (A). As we know that a functional series convergent in the mean or uniformly convergent can be integrated ***term-by-term***. Thus, suppose that series (A) converges uniformly to the function $f(x)$ on the interval $[-L, L]$. It remains convergent (in the same sense) if we multiply both sides of relation (A) by any continuous function. This property and the orthogonality of the system enable us to perform the operations given bellow and to determine the coefficients a_0, a_n, and b_n, $n = 1,2, \ldots$

■ To determine a_0 we can integrate Eq(A) termwise from $-L \; to \; L$, we obtain

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx = a_0 L$$

Thus,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\;dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

■ To determine a_n we multiply Eq(A) by $\cos\frac{m\pi x}{L}$ and integrate it with respect to *x* from $-L \; to \; L$. This results (by orthogonality conditions) in

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{a_0}{2}\int_{-L}^{L} \cos\frac{m\pi x}{L}\;dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (C)$$

Keeping in mind that *m* is a fixed positive integer ($m > 0$) wherease *n* ranges over the positive integers, it results from the orthogonality conditions that the only nonvanishing

term on the right side of Eq(C) is the one for which $m = n$ in the first summation. That is, the first and third integrals on the right-hand side of Eq(C) are always zero, but second integral is equal to L when $= n$; otherwise it also vanishes when $m \neq n$. Hence,

$$\underbrace{\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx}_{since\ m=n} = a_n \underbrace{\int_{-L}^{L}\left(\cos\frac{n\pi x}{L}\right)^2 dx}_{since\ m=n} = \frac{a_n}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx = a_n L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx\ , \qquad n = 1,2,3,\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (D)$$

Comparing to Eq(B), the formula of a_n can be written as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx\ , \qquad n = 0,1,2,3,\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (E)$$

By writing the constant term in Eq(A) as $\frac{a_0}{2}$, it is possible to compute all of the a_n from Eq(E) . Otherwise, a separate formula would have to be used for a_0.

■ To determine b_n we multiply Eq(A) by $\sin\frac{m\pi x}{L}$ and integrate it with respect to *x* from $-L\ to\ L$, and using the orthogonality conditions, we obtain

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$
$$+\sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

The first two integrals on the right-hand side are always zero, but the third integral is equal to L when $m = n$, otherwise it also vanishes when $m \neq n$. Hence

$$\underbrace{\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx}_{since\ m=n} = \frac{a_0}{2}(0) + a_n(0) + \frac{1}{2}b_n \underbrace{\int_{-L}^{L} 2\left(\sin\frac{n\pi x}{L}\right)^2 dx}_{since\ m=n}$$

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{2}b_n \int_{-L}^{L}\left(1-\cos\frac{2n\pi x}{L}\right)dx = b_n L$$

Thus,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\,,\ \ n = 1,2,3,\ldots \qquad \ldots\ldots\ldots\ldots\ldots (F)$$

Remarks:

(1) The numbers a_n and b_n in Eqs(E-F) are known as the Euler-Fourier formulas for the coefficients of the function $f(x)$ w.r.t. trigonometric system,

$$\frac{1}{2}, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}$$

(2) The trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$ whose coefficients are specified by the function $f(x)$ according to formulas in Eqs(E-F) is called the Fourier series of the function $f(x)$.

(3) It should be noted that the Fourier coefficients a_n and b_n in formulas *E* and *F* can be defined without imposing the requirement that series in Eq(A) is convergent. Indeed, it is sufficient that the function $f(x)$ be integrable on $[-L, L]$. Therefore, to every function $f(x)$ integrable on $[-L, L]$ there corresponds its Fourier series. This correspondence can be written in the form

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (G)$$

But in the general case when the function $f(x)$ satisfies only the condition that it is integrable on $[-L, L]$, the sign of correspondence ($\sim$) in relation (G) cannot be replaced by the sign of equality.

(4) In case the Fourier series of f converges uniformely, the definitions of Fourier constants suggest that its sum will be f, and that f is capable of a unique Fourier series expansion.

A host of questions arise but there are three major problems which prick the mind and which we shall try to answer in the subsequence sections.

(1) Is the Fourier series of a given (integrable) function f convergent for some or all values of x in $-\pi \leq x \leq \pi$?

(2) If it converges, does the Fourier series of f converges to ?

(3) When and how can a function be expressed as a Fourier series.

Definition: A function f is called **piecewise monotonic** on $[a, b]$ if there exists a partition of $[a, b]$ such that the function is monotonic (either monotone increasing or monotone decreasing) in each of the sub-interval. **piecewise continuous** functions are defined in the same way.

Definition: A function f is said to be **piecewise continuous** (***piecewise constant***) on $[a, b]$ if there are $x_0, x_1, x_2, \dots, x_n \ni a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and

(1) f is continuous (*constant*) in each subinterval (x_{j-1}, x_j) , $j = 1,2, \dots, n$,
(2) $f(x_0 + 0)$, $f(x_n - 0)$, and $f(x_j \pm 0)$, $j = 1,2, \dots, n-1$ exist.

Note that at each point of continuity of $f(x)$, we have $f(x-0) = f(x+0) = f(x)$, and thus $f(x-0)$ and $f(x+0)$ do not coincide only at a finite number of points at which $f(x)$ has the jumps $f(x+0) - f(x-0) \neq 0$.

Definition: A function $f(x)$ is said to be **piecewise smooth** on $[a, b]$ if it is a *piecewise continuous* function and if the derivative $f'(x)$ exists and is continuous everywhere on $[a, b]$ ***except*** possibly at a finite number of points at which finite right-hand and left-hand limits $f'(x+0) = \lim_{\substack{z \to 0 \\ z>0}} f'(x+z)$ and $f'(x-0) = \lim_{\substack{z \to 0 \\ z>0}} f'(x-z)$ exist.

It is also supposed that finite limits $f'(a+0)$ and $f'(b-0)$ exist at the end points *a* and *b* of $[a, b]$.

Formulation of Fundamental Theorem on Convergence of Fourier Series:

Theorem: If $f(x)$ is a ***piecewise smooth*** function on $[-L, L]$, its Fourier series converges at each point $x \in [-L, L]$ and the sum

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

of the series satisfies the following relations:

(1) $S(x) = f(x)$ if $x \in (-L, L)$ and $f(x)$ is continuous at the point *x*.

(2) $S(x) = \frac{f(x+0)+f(x-0)}{2}$ if $x \in (-L, L)$ and *x* is a point of discontinuity of $f(x)$.

(3) $S(-L) = S(L) = \frac{f(-L+0)+f(L-0)}{2}$.

Remark: Note that, if $f(x)$ is continuous at a point *x*, $x \in (-L, L)$, we have

$f(x + 0) - f(x - 0) = f(x)$, and consequently,

$$\frac{f(x + 0) + f(x - 0)}{2} = \frac{2f(x)}{2} = f(x)$$

Therefore,

$$S(x) = \frac{f(x + 0) + f(x - 0)}{2}$$

Which holds for every interior point *x* of $[-L, L]$.

Example 1.1.1: Let us take the function $f(x) = x$ regards as being defined on the interval $[-L, L]$ and expand it into a Fourier series.

Solution: We need to determine the coefficients a_0 , $a_n, and\ b_n$. By applying the relations *B*, *E*, and *F* respectively, we obtain $a_0 = 0$, $a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\ dx$.

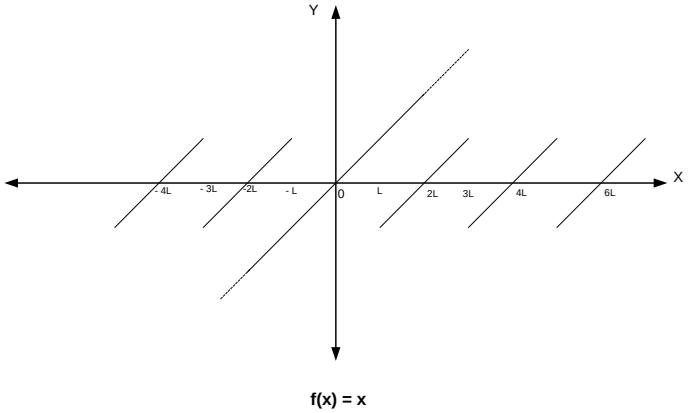
Integrating by parts, we obtain

$$b_n = \frac{2}{L}\int_0^L x \sin \frac{n\pi x}{L}\ dx = \frac{2}{L}\left\{ \left(-\frac{xL}{n\pi} cos \frac{n\pi x}{L} \right)\Big|_{x=0}^{L} + \frac{1}{n\pi}\int_0^L \cos \frac{n\pi x}{L}\ dx \right\} = \frac{2L}{n\pi}(-1)^{n+1}$$

Consequently, according to the convergence theorem, we have

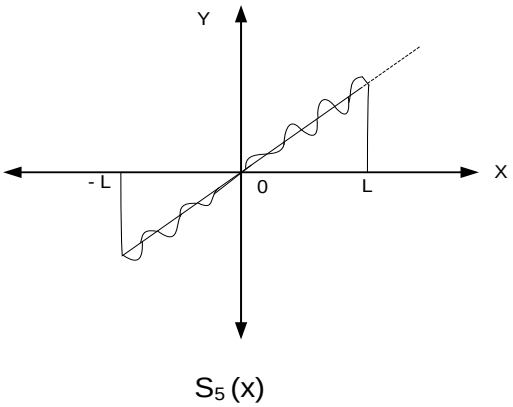
$$S(x)=\frac{2L}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{L}=\begin{cases}\dfrac{f(x+0)+f(x-0)}{2}=x & for\ x\in(-L,L)\\ \dfrac{f(-L+0)+f(L-0)}{2}=\dfrac{-L+L}{2}=0 & for\ x=\pm L\end{cases}$$

The graph of $f(x)=x$ and $S(x)$ are shown bellow.



f(x) = x

Fig.1



$S_5(x)$

Fig.2

The function $S(x)$ is periodic with period $2L$, and $S(x)=x$ only for $x\in(-L,L)$. At the points $=(2k+1)L$, $k=0,\pm1,\pm2,\ldots$, the function $S(x)$ has discontinuities, its left-hand and right-hand limits being respectively equal to $f[(2k+1)L-0]=-1$ and $f[(2k+1)L+0]=1$, $k=0,\pm1,\pm2,\ldots$ at these points.

At every point $x=(2k+1)L$, $k=0,\pm1,\pm2,\ldots$, the function $S(x)$ assumes the value which is the left-sum of the left-hand and right-hand limits (i.e. equal to zero). In Fig.2, we see the graph of the partial sum $S_5(x)$, where

$$S_5(x)=\frac{2L}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{L}$$

Constructed on $[-L,L]$.

Remark: In order to simplify the computations of the coefficients $a_0\ , a_n, and\ b_n$, we will give some properties of "**Even** and **Odd**" functions.

Even and Odd Functions:

Definition: A function $f(x)$ is said to be **even** if $f(-x)=f(x)$, $x\in D_f$.

Definition: A function $f(x)$ is said to be **odd** if $f(-x)=-f(x)$, $x\in D_f$.

Example: the functions $x^2,\ x^4+x^2+1,\ e^x+e^{-x},\ cosx,\ secx$ are all ***even*** functions of x, while $x,\ x^3,\ x^5+2x^3+3,\ sinx,\ cscx,\ tanx$ are all ***odd*** functions. However, $f(x)=e^x$ ***neithe***r odd ***nor*** even function.

Note that, graphically, ***an even function is symmetrical about the y-axis***, and ***an odd function is symmetrical about the origin***. In addition, the product of two even or two odd functions will be an even function, while the product of an even function and an odd function will be an odd function.

1.2 Fourier Series for Even and Odd Functions

Lemma: If *f* is integrable on every finite interval, then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & when\ f(x)\ is\ an\ odd\ function, \quad \forall a>0 \\ 2\int_{0}^{a} f(x)dx & when\ f(x)\ is\ an\ even\ function, \quad \forall a>0 \end{cases}$$

Definition: Let f be integrable (not necessarily periodic) on the interval $[-L,L]$. The Fourier series of f is the trigonometric series,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

where the coefficients $a_0\ , a_n, and\ b_n$ are given by Eqs(*B*, *E*, and *F*), respectively. In that case, we write

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(G)$$

Remark: This definition does not imply that the series (G) converges to *f* or that *f* is periodic. In addition, this definition and the above Lemma imply that if f is **even** in $[-L,L]$, then the coefficient $b_n=0$ (since $f(x)\sin x$ is an ***odd*** function), and hence the Fourier series of f has the form

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}\ \ldots\ldots\ldots\ldots\ldots\ldots(H)$$

and if f is ***odd***, then the coefficients $a_0=0$ (since $f(x)$ is an ***odd*** function), and $a_n=0$ (since $f(x)\cos x$ is an ***odd*** function), and hence the Fourier series of f has the form

$$f(x)\sim\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \ \ldots\ldots\ldots\ldots\ldots\ldots (I)$$

The series (H) and (I) are called the Fourier cosine series and Fourier sine series, respectively.

If $L=\pi$, then the Fourier series (G), ((H) and (I)) transforms to

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \ \ldots\ldots\ldots\ldots\ldots\ldots (J)$$

where the coefficients $a_0, a_n, and\ b_n$ are given by Eqs(*B*, *E*, and *F*), with $L=\pi$.

Example 1.2.1: Let us take the function $f(x)=x^2$ on $[-L,L]$. This function being ***even***, then we have $b_n=0$. The coefficients a_0, a_n will be computed as

$$a_0=\frac{2}{L}\int_0^L x^2\,dx \quad and \quad a_n=\frac{2}{L}\int_0^L x^2\cos\frac{n\pi x}{L}\,dx, \qquad n=1,2,3,\ldots$$

After computing the coefficients a_0, a_n, we find by the convergent theorem, that

$$S(x)=\frac{L^2}{3}+\frac{4L^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\frac{k\pi x}{L}=x^2 \ \ for\ x\in[-L,L] \ \ldots\ldots\ldots\ldots (1)$$

The function $S(x)$ is continuous everywhere including the points $(2k+1)\pi$, $k=0,\pm1,\ldots$ since for the function $f(x)=x^2$ we have $f(-L+0)=f(L-0)=L^2$.

If we apply this expansion to the interval $[-\pi,\pi]$, that is, for $L=\pi$, equality (1) takes the form

$$x^2=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx \ , \qquad x\in[-\pi,\pi] \ \ldots\ldots\ldots\ldots (2)$$

Substituting $x=0$ into Eq(2), we obtain the useful relation

$$0=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\rightarrow-\frac{\pi^2}{3}=4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\rightarrow-\frac{\pi^2}{12}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}$$

Thus,

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + (-1)^{k-1}\frac{1}{k^2} + \cdots \qquad \ldots\ldots\ldots(3)$$

Example 1.2.2: Let $f(x)$ be defined on $[-L, L]$ by the relations

$$f(x) = \begin{cases} c_1 & for\ x \in (-L, 0) \\ c_2 & for\ x \in (0, L) \end{cases}$$

Then,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\left[\int_{-L}^{0} c_1\,dx + \int_{0}^{L} c_2\,dx\right] = c_1 + c_2$$

and

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\left[\int_{-L}^{0} c_1\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} c_2\cos\frac{n\pi x}{L}\,dx\right], \qquad n = 1,2,\ldots$$

Consequently,

$$b_n = \begin{cases} 0 & for\ n\ is\ even \\ \dfrac{2(c_2 - c_1)}{n\pi} & for\ n\ is\ odd \end{cases}$$

Therefore, by the convergent Theorem, we obtain

$$S(x) = \frac{(c_1 + c_2)}{2} + \frac{2(c_2 - c_1)}{\pi}\left\{sin\frac{\pi x}{L} + \frac{1}{3}sin\frac{3\pi x}{L} + \frac{1}{5}sin\frac{5\pi x}{L} + \cdots\right\}$$

$$= \begin{cases} c_1 & for\ x \in (-L, 0) \\ c_2 & for\ x \in (0, L) \\ \dfrac{(c_1 + c_2)}{2} & for\ x = \pm L,\ \ x = 0 \end{cases}$$

The graph of $S(x)$ is shown bellow (Fig.1) bellow.

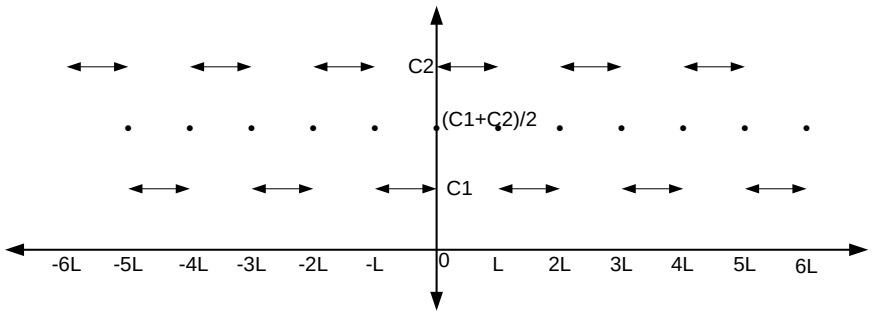


Fig.1: S(x)

In Fig. 2, we see the graph of the partial sums $S_1(x)$ and $S_3(x)$ for the case $c_1 = -1, c_2 = 1$. In this case, the series turns into

$$S(x) = \frac{4}{\pi}\left\{sin\frac{\pi x}{L} + \frac{1}{3}sin\frac{3\pi x}{L} + \frac{1}{5}sin\frac{5\pi x}{L} + \cdots\right\}$$
$$= \begin{cases} -1 & for\ x \in (-L, 0) \\ 1 & for\ x \in (0, L) \\ 0 & for\ x = \pm L\ ,\ \ x = 0 \end{cases}$$

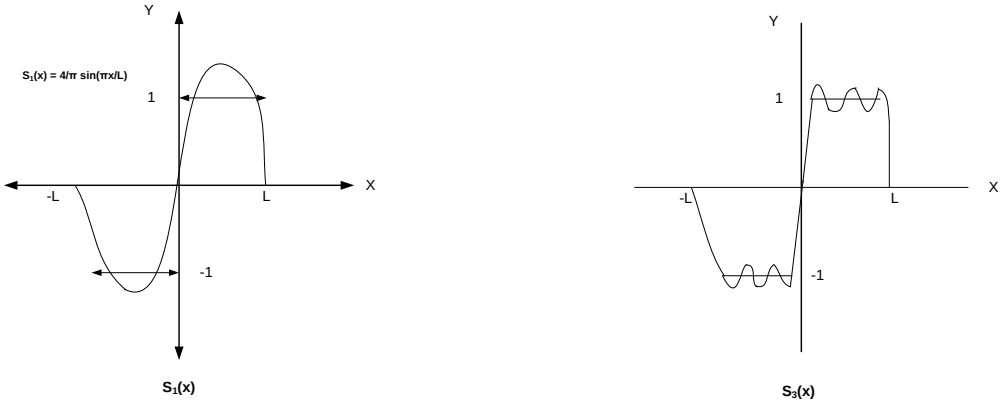


Fig. 2

Theorem: If f is bounded and integrable on $[-\pi, \pi]$ and if $a_n\ , b_n$ are its Fourier coefficients, then $\sum_{n=1}^{\infty}\left({a_n}^2 + {b_n}^2\right)$ converges.

Proof: In fact, the integral $\int_{-\pi}^{\pi}[f - \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx]\, dx \geq 0$ as its integral is never negative. Note that,

$$\int_{-\pi}^{\pi}\left[f - \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]^2 dx = \int_{-\pi}^{\pi} f^2\, dx - 2\sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-\pi}^{\pi} f \cos nx\, dx}_{=a_n\pi}\right]$$

$$-2\sum_{n=1}^{\infty}\left[b_n \underbrace{\int_{-\pi}^{\pi} f\sin nx\,dx}_{=b_n\pi}\right]+\int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx\right]^2 dx$$

$$=\int_{-\pi}^{\pi} f^2\,dx-2\pi\sum_{n=1}^{\infty}{a_n}^2-2\pi\sum_{n=1}^{\infty}{b_n}^2+\pi\sum_{n=1}^{\infty}{a_n}^2+\pi\sum_{n=1}^{\infty}{b_n}^2$$

$$=\int_{-\pi}^{\pi} f^2\,dx-\pi\sum_{n=1}^{\infty}\left({a_n}^2+{b_n}^2\right)\geq 0\rightarrow\int_{-\pi}^{\pi} f^2\,dx\geq\pi\sum_{n=1}^{\infty}\left({a_n}^2+{b_n}^2\right)$$

Then,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx\geq\sum_{n=1}^{\infty}\left({a_n}^2+{b_n}^2\right)\rightarrow\sum_{n=1}^{\infty}\left({a_n}^2+{b_n}^2\right)\leq\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$$

Thus, $\sum_{n=1}^{\infty}\left({a_n}^2+{b_n}^2\right)$ converges. ■

The Main Theorem: We are now in a position to take up the main problem. Our approach will be to write down the partial sum sequence of the Fourier series of the function f and lay down a set of **sufficient conditions** so that the sequence (and consequently the Fourier series) converges to f. That will answer ***all the three questions raised earlier***, that a function f satisfying these conditions will generate a Fourier series which converges and has a sum f.

A large number of sets of sufficient conditions for the purpose have been found, of which we shall use only one, "**Dirichlet's criterion**", the great generality of which renders it sufficient for most purposes. It was given by Dirichlet and was the first set of exact conditions of convergence in the theory of Fourier series.

Conditions for Fourier Expansion:

Dirichlet has formulated certain conditions known as ***Dirichlet conditions*** under which certain functions possess valid Fourier expansions.

A function $f(x)$ has a valid Fourier series expansion of the form,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\,cos\,nx+b_n\,sin\,nx$$

Where a_0 , a_n , $and\ b_n$ are constants, provided
(i) $f(x)$ is well defined and single-valued, except possibly at a finite number of points in the interval of definition.
(ii) $f(x)$ has a finite number of finite discontinuities in the interval of definition.
(iii) $f(x)$ has at most a finite number of maxima and minima in the interval of definition.

Note that, the above conditions are **sufficient but Not necessary**. The following theorem states the Dirichlet's criterion.

Theorem (Dirichlet's criterion): If a function f is bounded and integrable on $[-\pi, \pi]$, and piecewise monotonic on $[-\pi, \pi]$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\lambda + b_n \sin n\lambda = \begin{cases} \dfrac{f(\lambda -) + f(\lambda +)}{2} & for\ \lambda \in (-\pi, \pi) \\ \dfrac{f(\pi -) + f(\pi +)}{2} & for\ \lambda = \pm\pi \end{cases}$$

Proof: Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ be the Fourier series of f, and λ a point of $[-\pi, \pi]$. The *mth* partial sum at the point λ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\lambda + b_n \sin n\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{m} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,(\cos nx \cos n\lambda + \sin nx \sin n\lambda)dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left[1 + 2\sum_{n=1}^{m} \cos n(x - \lambda)\right] dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,(x + \lambda) \left[1 + 2\sum_{n=1}^{m} \cos nx\right] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,(x + \lambda) \frac{\sin\left(m + \frac{1}{2}\right)x}{\sin\frac{1}{2}x} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} f(x)\,(x + \lambda) \frac{\sin\left(m + \frac{1}{2}\right)x}{\sin\frac{1}{2}x} dx + \frac{1}{2\pi} \int_{0}^{\pi} f(x)\,(x + \lambda) \frac{\sin\left(m + \frac{1}{2}\right)x}{\sin\frac{1}{2}x} dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} f(t)\,(-2t+\lambda)\frac{sin(2m+1)t}{sin\,t}\,dt + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} f(t)\,(2t+\lambda)\frac{sin(2m+1)t}{sin\,t}\,dt$$

Proceeding to limits when $n \to \infty$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,cos\,n\lambda + b_n\,sin\,n\lambda = \frac{1}{\pi}\left[\frac{1}{2}\pi\,f(\lambda-) + \frac{1}{2}\pi\,f(\lambda+)\right] = \frac{f(\lambda-)+\,f(\lambda+)}{2}\,,\lambda \in [-\pi,\pi]$$

Thus, the Fourier series of a (***periodic***) function f which is bounded, integrable and piecewise monotonic on $[-\pi,\pi]$ converges to $\frac{f(\lambda-)+f(\lambda+)}{2}$ at a point $\lambda \in (-\pi,\pi)$, and (using the periodicity of f) to $\frac{f(\pi-)+f(-\pi+)}{2}$ at the ends $\pm\pi$.

Corollary: At a point η of **continuity** of f, the sum of the Fourier series of f is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\,cos\,n\eta + b_n\,sin\,n\eta = \frac{f(\eta-)+\,f(\eta+)}{2} = \frac{f(\eta)+\,f(\eta)}{2} = f(\eta)$$

Thus, for a function f satisfying the conditions of the above Theorem, the sum of the Fourier series is actually $f(x)$ at all points x whose f is **continuous**, while at a points of (finite) **discontinuities**, the sum of the series is $\frac{f(x-)+f(x+)}{2}$.

Example 1.2.3: Find the Fourier series of the periodic function f with period 2π, defined as follows:

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

What is the sum of the series at $= 0, \pm\pi, 4\pi, -5\pi$?

Solution: The function is bounded, integrable, and piecwise monotonic on $[-\pi,\pi]$. Let us determine the Fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x\,dx\right] = \frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\,sin\,nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} sin\,nx\,dx\right] = \frac{1}{n\pi}\left(\frac{cos\,nx}{n}\Big|_0^{\pi}\right) = \begin{cases} 0 & if\ \ n\ \ even \\ -\dfrac{2}{\pi n^2} & if\ \ n\ \ odd \end{cases} \ldots\ldots..(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,sin\,nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\ sin\,nx\,dx\right] = \frac{1}{\pi}\int_0^{\pi} x\ sin\,nx\,dx$$

$$= -\frac{\cos n\pi}{n} = \begin{cases} -\dfrac{1}{n} & if\ \ n\ \ even \\ \dfrac{1}{n} & if\ \ n\ \ odd \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now; in $[-\pi, \pi]$, the points $\pm\pi$ are the only points of discontinuity of f. Therefore, at all points of the interval other than $\pm\pi$, we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\} + \left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}$$

At $x = 0$, where f is continuous, we get

$$f(0) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} = 0$$

Then,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Because of periodicity, value of the Fourier series at $x = 4\pi$ is same as at $x = 0$.
At $x = \pm\pi$, points of discontinuity of f , the sum of the series is

$$\frac{1}{2}\left[\underbrace{f(\pi -)}_{=\pi} + \underbrace{f(\pi +)}_{=0}\right] = \frac{\pi}{2}$$

Then,

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

Then,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Again because of periodicity, value of the Fourier series at $x = -5\pi$ is same as at $x = \pm\pi$. Fig.1 in the following, gives the shape of the function f.

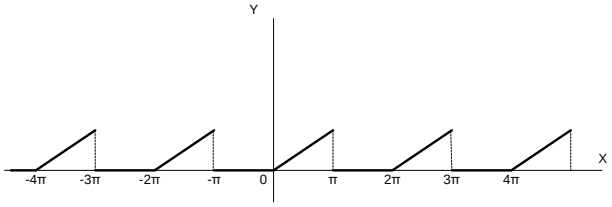


Fig.1

Example 1.2.4: Determine the Fourier series expansion of the function

$$f(x) = \pi^2 - x^2 \quad for\ -\pi \le x \le \pi$$

then deduce a series expansion for π^2 using the Fourier series expansion of $f(x)$ in $[-\pi, \pi]$

Solution: We will compute the Fourier coefficients $a_0, a_n, and\ b_n$, $n = 1,2,3, \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)dx = \frac{4}{3}\pi^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\cos nx\, dx = \pi\int_{-\pi}^{\pi}\cos nx\, dx - \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\, dx$$

$$a_n = \pi\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\left(\frac{2x\cos nx}{n^2} + \frac{(n^2x^2 - 2)\sin nx}{n^3}\right)\Big|_{-\pi}^{\pi} = (-1)^{n+1}\frac{4}{n^2} \quad \ldots.(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\sin nx\, dx = \pi\int_{-\pi}^{\pi}\sin nx\, dx - \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\, dx = 0 \ldots\ldots(3)$$

Using integration by parts, the result could, of course, have been obtained by direct evaluation, though the argument would have been more tedious.

Instead of evaluating this integral directly, let us divide the interval of integration and rewrite the result as the sum of two integrals.
First we write

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi^2 - x^2)\,sin\,nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - x^2)\,sin\,nx\,dx$$

and then setting $x = -z$ in the first integral, this becomes

$$b_n = -\frac{1}{\pi}\int_{\pi}^{0}(\pi^2 - z^2)\,sin(-nz)\,dz + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - x^2)\,sin\,nx\,dx$$

However, $sin(-nz) = -sin(nz)$, and the minus sign in front of the first integral may be utilized to reverse the order of the limits of integration, so that finally we arrive at

$$b_n = -\frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - z^2)\,sin(nz)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - x^2)\,sin\,nx\,dx\,, n = 1,2,3,\ldots$$

As the variable in a definite integral is only a dummy variable, we may replace *z* by *x* in the first integral to deduce that $b_n = 0 \quad \forall n$.

Thus, the ***Fourier series*** of $f(x)$ is given by

$$f(x) = \frac{2}{3}\pi^2 + 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots + \frac{(-1)^{n+1}}{n^2}\cos nx + \cdots\right) \ldots\ldots (4)$$

Now; we will deduce a series expansion for π^2 using ***Fourier series*** of $f(x)$ is given in Eq(4). Note that, the function $f(x) = \pi^2 - x^2\ ,\ x \in [-\pi, \pi]$ is ***continuous*** in $[-\pi, \pi]$, then by theorem, its Fourier series converges to $f(x)$ at every point in that interval. Hence using the Fourier series of Eq(4) with $x = \frac{\pi}{2}$, we obtain

$$\pi^2 - \frac{\pi^2}{4} = \frac{2}{3}\pi^2 + 4\left(\cos\frac{\pi}{2} - \frac{1}{2^2}\cos\pi + \frac{1}{3^2}\cos\frac{3\pi}{2} - \cdots\right)$$

which, after simplification, gives the result

$$\pi^2 = 48\left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \cdots\right) \ldots\ldots\ldots\ldots\ldots (5)$$

The relationship between the Fourier series representation of $f(x) = \pi^2 - x^2$ in the fundamental interval $[-\pi, \pi]$, the periodic extension it assigns to $f(x)$ outside the fundamental interval, and the actual behavior of $f(x)$ both inside and outside the fundamental interval are illustrates in the ***following Figure***. The full curve denotes both the functional behavior of $f(x)$ and that of its Fourier series in the fundamental interval, the dotted curve denotes the periodic extension of $f(x)$, and the chaindotted curve denotes the actual behavior of $f(x)$ outside the fundamental interval. Of course, it still remains for us to justify our assertion that the Fourier series converges to $f(x)$ in $[-\pi, \pi]$, but it certainly does so when $x = 0$ and $x = \pm\pi$. This follows by employing the standard results

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

to evaluate the Fourier series at those points.

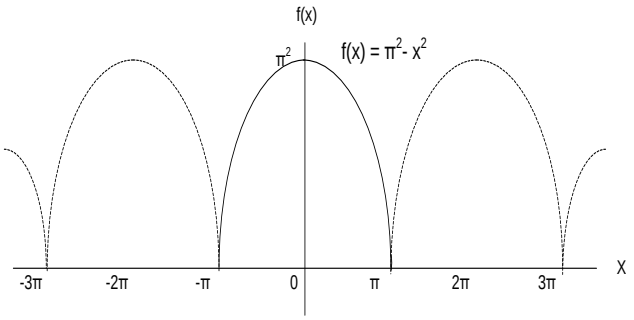
Let us define the ***mth*** partial sum $S_m(x)$ of this Fourier series to be

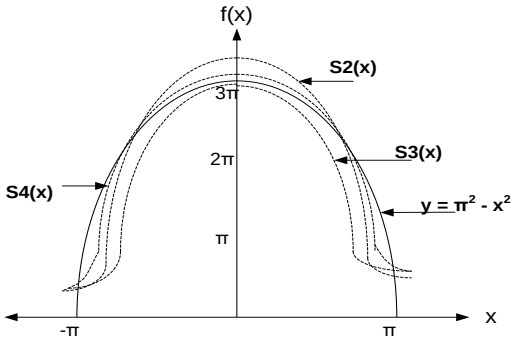
$$S_m(x) = \frac{2}{3}\pi^2 + 4\sum_{n=1}^{m-1} \frac{(-1)^{n-1}}{n^2} \cos nx$$

Then, when working numerically with the Fourier series, the function will need to be approximated by a partial sum. The behavior of the second, third, and fourth partial sums

$$S_2(x) = \frac{2}{3}\pi^2 + 4\cos x\,,$$
$$S_3(x) = \frac{2}{3}\pi^2 + 4cosx - cos2x\,,$$
$$S_4(x) = \frac{2}{3}\pi^2 + 4cosx - cos2x + \frac{4}{9}cos3x$$

is shown in the ***following Figure***, and they certainly suggest the convergence of $S_n(x)$ to $f(x)$ over $[-\pi, \pi]$.





Exercise 1.2.5: Deduce a series expansion for π^2 using the Fourier series expansion of the function $f(x)$ given in ***Example 1.2.3*** .

Solution: The function $f(x)$ is discontinuous at the point $x = \pi$, and by theorem and the periodic extension of $f(x)$ beyond the interval $[-\pi, \pi]$, the Fourier series converges to

$$\frac{1}{2}\left[\underbrace{f(\pi -)}_{=\pi} + \underbrace{f(\pi +)}_{=0}\right] = \frac{\pi}{2}$$

Hence, setting $x = \pi$ and $f(\pi) = \frac{\pi}{2}$ in the given Fourier series, we obtain

$$\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$$

Whence

$$\pi^2 = 8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Exercise 1.2.6: Expand in a series of sines and cosines of multiple angles of *x* , the periodic function with period 2π defined as

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Also calculate the sum of the series at $x = 0, \frac{\pi}{2}, \pm\pi$.

Solution: The function, following Figure, is piecewise monotonic, bounded and integrable. Let us compute its Fourier coefficients $a_0, a_n, and\ b_n$, $n = 1,2,3, \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_{0}^{\pi}(1)dx\right] = 0 \quad \ldots\ldots\ldots\ldots(1)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right] = \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n\ even \\ \dfrac{4}{n\pi} & n\ odd \end{cases} \ldots.(3)$$

The function is continuous at all points of $[-\pi, \pi]$ except $\pm\pi$. Thus,

$$f(x) = \frac{4}{\pi}\left\{\frac{sin\,x}{1} + \frac{sin\,3x}{3} + \frac{sin\,5x}{5} + \cdots\right\}$$

which holds at all points with the exception of the discontinuities at $\pm\pi$.
At $x = \pm\pi$, the sum of the series is

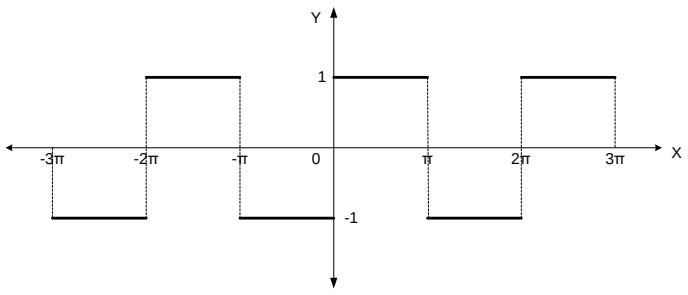
$$\frac{1}{2}\left[\underbrace{f(\pi-)}_{=1} + \underbrace{f(\pi+)}_{=-1}\right] = 0$$

At $x = \frac{\pi}{2}$, a point of continuity,

$$f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left\{\frac{sin\frac{\pi}{2}}{1} + \frac{sin\,3\frac{\pi}{2}}{3} + \frac{sin\,5\frac{\pi}{2}}{5} + \cdots\right\} \Rightarrow 1 = \frac{4}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\}$$

Thus,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$



Exercise 1.2.7: Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

Hence show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi}(k)dx\right] = 0 \quad \ldots\ldots\ldots\ldots(1)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right] = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi}(k)\sin nx\,dx\right] = \frac{k}{\pi}\left[\left(\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} - \left(\frac{\cos nx}{n}\right)\Big|_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[\left(\frac{1-(-1)^n}{n}\right) - \left(\frac{(-1)^n - 1}{n}\right)\right] = \frac{k}{\pi}\left[\frac{2-2(-1)^n}{n}\right] = \begin{cases} 0 & n\ even \\ \dfrac{4k}{n\pi} & n\ odd \end{cases} \ldots\ldots(3)$$

Thus,

$$f(x) = \sum_{n=1}^{\infty}\frac{4k}{n\pi}\sin nx \quad where\ n\ is\ odd$$

$$f(x) = \frac{4k}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}$$

Which holds at all points with the exception of the discontinuities at $\pm\pi$.
At $x = \pm\pi$, the sum of the series is

$$\frac{1}{2}\left[\underbrace{f(\pi-)}_{=k} + \underbrace{f(\pi+)}_{=-k}\right] = 0$$

At $x = \frac{\pi}{2}$, a point of continuity,

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left\{\frac{sin\frac{\pi}{2}}{1} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right\} \Rightarrow 1 = \frac{4}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\}$$

Thus,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

There are different approaches if the function f is defined on an asymmetric interval $[0, L]$ with arbitrary $L > 0$. These approaches will be introduced in the following subsection.

1.3 Fourier Cosine and Sine Series on a Half-Interval (Half-Range Expansions)

In many applications we are interested in representing a function $f(x)$ that is defined only in a finite interval, say $0 < x < L$ by a Fourier series. **Since f is clearly not periodic**, the results of the previous sections are not readily applicable. **Our goal** in this section is to show how we can represent f by a Fourier series, after extending it to a periodic function.

Suppose *f* is defined on the interval [0,L]. By considering ***even*** or ***odd*** extensions of *f* , we can expand *f* as a cosine or sine series. To express *f* as a **cosine** series, we consider the ***even extension*** of *f*:

■ *Even extension*. Define a function $g(x)$ on the interval [-L,L] as

$$g(x) = f_e(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0 \end{cases}$$

Then $g(x)$ is an **even** function defined on [-L,L] and its Fourier (cosine) series

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \;\; ; \; -L \le x \le L; \ldots\ldots\ldots\ldots (K_1)$$

represents *f* on $[0, L]$, where $a_0 \; and \; a_n$ are given as

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \;\; ; and \;\; a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Since $f_e(x) = f(x)$ for $0 \le x \le L$, the integral formulas only involves $f(x)$ rather than $f_e(x)$ and so $Eq(K_1)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \; ; \quad 0 \le x \le L \quad \ldots\ldots\ldots\ldots\ldots (K_2)$$

Likewise, to express *f* as a sine series, then we consider the ***odd extension*** of *f*:

■ *Odd extension*. Define a function $h(x)$ on the interval [-L,L] as

$$h(x) = f_o(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases}$$

Then $h(x)$ is an **odd** function defined on [-L,L] and its Fourier (sine) series

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \;\; ; \; -L \le x \le L; \ldots\ldots\ldots\ldots (K_3)$$

represents f on $[0, L]$, where b_n is given as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

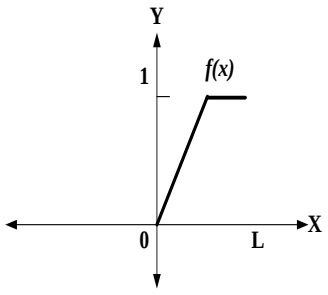
Since $f_o(x) = f(x)$ for $0 \le x \le$ L, the integral formulas only involves $f(x)$ rather than $f_e(x)$ and so $Eq(K_3)$ becomes,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \;\; ; \; -L \le x \le L; \ldots\ldots\ldots\ldots (K_4)$$

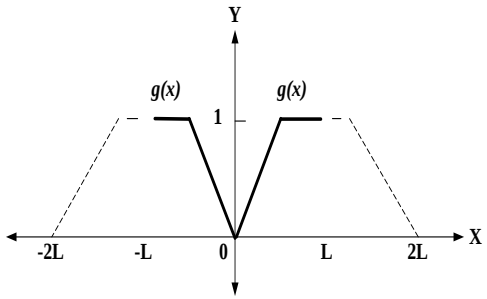
Each of the series in Eqs.(K_2 and K_4) converges to $f(x)$ on $(0, L)$ at the points of continuity of $f(x)$.

Thus, simply by imagining the appropriate extension of a function f defined only from **o** to **L**, we can obtain expansions representing f (**at least where f is continuous**) and containing only cosine terms or only sine terms, as we please. Moreover, the formulas for the coefficients in these expansions involve only the values of f on the original interval from **o to L** and do not depend in any way upon the extensions we used to obtain those formulas. Such series Eqs.(K_2 and K_4) are known as **half-range expansions** of f. They are two different series representations of the same function on the interval $[0, L]$. The constructed function g is **even and 2L-periodic**, and the function h is **odd and 2L-periodic**. Both functions agree with f on $[0, L]$, which justifies calling them extensions of f.

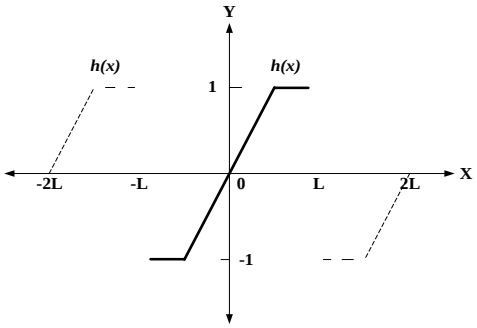
The following Figure shows the extensions of f on $(0, L)$ (in view of the previous remarks, we will not worry about the definition of the extensions at the points $0, \pm L, \pm 2L, \ldots$).



(a) $f(x)$, $0 < x < L$



(b) *Even 2-periodic extension,* f.



(c) *Odd 2-periodic extension,* f.

Example 1.3.1: Let $f(x) = x$ on the interval $[0, L]$. If $f(x)$ is extended "in odd fashion" we arrive at the function $f(x) = x$ defined on the interval $[-L, L]$ whose Fourier series has already been studied (see Example 1.1.1).

Now, extending $f(x)$ "in even fashion" we obtain the function $f(x) = |x|$ on the interval $[-L, L]$. Expanding $f(x) = |x|$ into a cosine series on the interval $[0, L]$ we obtain by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{L} \; ; \quad 0 \le x \le L \quad \ldots\ldots\ldots\ldots\ldots (K_2)$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \;\; ; and \;\; a_n = \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx$$

the relation

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{L} \; ; \quad 0 \le x \le L$$

where

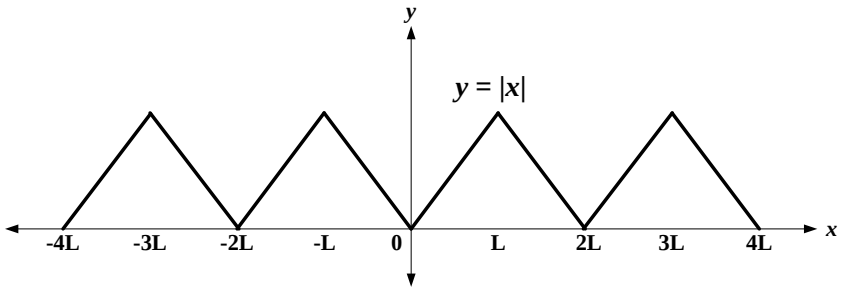
$$a_0 = \frac{2}{L}\int_0^L xdx = L \;\; ; and \;\; a_n = \frac{2}{L}\int_0^L x\, cos\frac{n\pi x}{L}dx = \frac{2L}{\pi^2 n^2}[(-1)^n - 1] = \begin{cases} 0 & n \; even \\ \dfrac{-4L}{\pi^2 n^2} & n \; odd \end{cases}.$$

Consequently,

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left\{\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \cdots\right\}, \qquad 0 \le x \le L \qquad (1)$$

The validity of equality (1) for $x = 0$ and $x = L$ can easily be established if we regard (1) as Fourier's series of the even function $f(x) = |x|$ defined on the interval $[-L, L]$. Such an

interpretation of series (1) enables us to construct the graph of its sum for any values of x (see the following Figure).



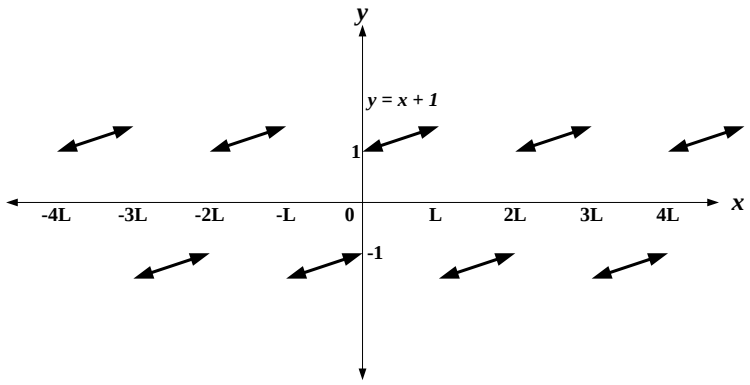
Generally, when an arbitrary piecewise continuous and piecewise smooth function $f(x)$ defined on an interval $[0, L]$ is extended in **even fashion** to the interval $[-L, 0]$ **we always have**

$$f(-0) = f(+0) \; and \; f(-L+0) = f(L-0) \qquad (2)$$

Consequently, the sum of the corresponding Fourier series is **continuous** at the points $x = 0$ and $= \pm L$, and its values at these points are, respectively, equal to $f(+0) = f(-0)$ and $f(-L+0) = f(L-0)$. If, in addition, the original function $f(x)$ is continuous at the end points of the interval $[0, L]$, that is $f(+0) = f(0)$ and $f(L-0) = f(L)$, the sum of its cosine series is equal to $f(x)$ at these end points.

On the contrary, if we extend a function $f(x)$ (originally defined for $0 \le x \le L$) in odd fashion to the interval $-L \le x \le 0$, the sum of the corresponding **sine** series may have discontinuities at the points $x = 0$ and $x = \pm L$ despite the continuity and smoothness of $f(x)$ on the interval $0 \le x \le L$. **In the following Figure**, we see the graph of the sum of the F.S. corresponding to the function $f(x) = x + 1$ originally defined for $0 \le x \le L$ and then extended in odd fashion. In the general case, when a function $f(x)$ is extended as an odd function from an interval $[0, L]$ to the interval $[-L, 0]$ we always have $f(-0) = -f(+0)$ and $f(-L+0) = -f(L-0)$. But for the sum of the corresponding Fourier series to be continuous at the points $x = 0$ and $x = \pm L$ it is necessary that the relations $f(-0) = f(+0)$ and $f(-L+0) = f(L-0)$ hold, and hence this sum is continuous only if

$$f(+0) = 0 \quad and \quad f(L-0) = 0 \qquad (3)$$



Example 1.3.2: Expand the function $f(x) = sinx$ in terms of *F.S.* consists of only cosines on $(0, \pi)$.

Solution: We know that it cannot be obtained *F.S.* contains only cosines unless in the case of ***even*** functions. Therefore, ***we have to find*** *F.S.* of the ***periodic even extension*** $g(x)$ of (x) . Fourier series of $g(x)$ converges to $f(x)$ at each point of continuity of $f(x)$ since $g(x) = f(x)$ on $(0, \pi)$. That is, *F.S.* of $g(x)$ will converge to $f(x) = sinx$ on $(0, \pi)$.

Here, $L = \pi$, and $b_n = 0$ since $g(x)$ is an even function. Also,

$$a_n = \frac{2}{L}\int_0^L g(x)cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi sin\,x\,cos\,nx\,dx = \frac{1}{\pi}\int_0^\pi sin(x+nx) + sin(x-nx)\,dx$$

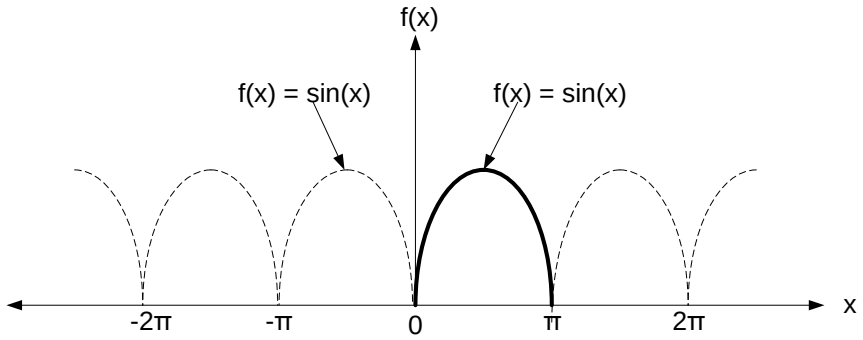
Then,

$$a_n = \frac{-2(1+\cos n\pi)}{\pi(n^2-1)}\;;\quad n \neq 1$$

$$If\ n = 1;\ \ we\ have\ a_1 = \frac{2}{\pi}\int_0^\pi sinx\ cosx\ dx = \frac{2}{\pi}\frac{sin^2x}{2}\Big|_0^\pi = 0\ .\ Therefore,$$

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+cosn\pi}{n^2-1}\cos nx = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{cos2x}{2^2-1} + \frac{cos4x}{4^2-1} + \frac{cos6x}{6^2-1} + \cdots\right).$$

The following Figure shows the periodic even extension of $f(x)\,, 0 < x < \pi$. We extend the function f from $0\ to - \pi$ by reflecting it in the vertical axis (y-axis), so that $f(-x) = f(x)$, the original function together with its extension is even. Hence the Fourier expansion of the periodic continuation g will contain only cosine terms (including, of course, $\frac{a_0}{2}$).



periodic even extension of f(x)

Example 1.3.3: Expand the function $f(x) = x$ in terms of *F.S.* consists of only cosines on $(0,2)$.

Solution: We know that it **cannot** be obtained *F.S.* contains only cosines **unless in the case of even functions**.

Therefore, we have to find *F.S.* of the periodic even extension $g(x)$ of $f(x)$. Fourier series of $g(x)$ converges to $g(x)$ at each point of continuity of $g(x)$ since $g(x) = f(x)$ on $(0,2)$. That is, *F.S.* of $g(x)$ will converge to $f(x) = x$ on $(0,2)$.

Here, $L = 2$, since $g(x)$ is an even function , then $b_n = 0$. Also,

$$a_n = \frac{2}{L}\int_0^L g(x)cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\,cos\left(\frac{n\pi x}{2}\right)dx = \left(x\frac{2}{n\pi}sin\frac{n\pi x}{2} - \frac{-4}{n^2\pi^2}cos\frac{n\pi x}{2}\right)\Big|_0^2$$

$$a_n = \frac{-4(cos(n\pi) - 1)}{n^2\pi^2}\ ;\quad n \neq 0$$

$$If\ n = 0;\ \ we\ have\ a_0 = \int_0^2 x\ \ dx = 2\ \ .Therefore,$$

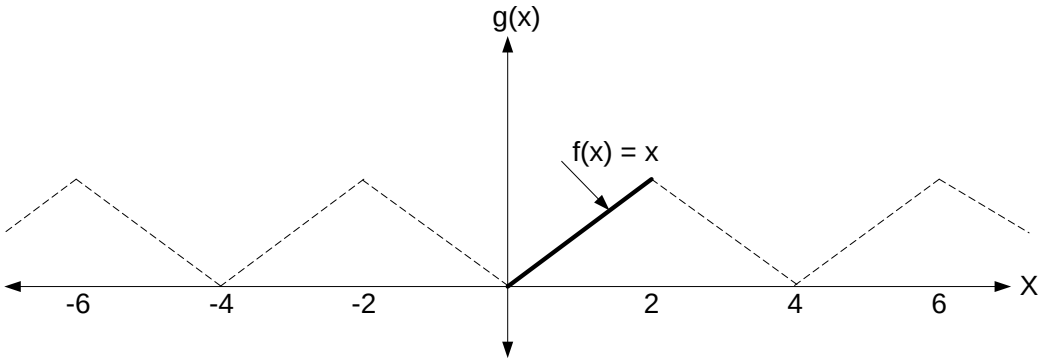
Then,

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4(cos\ (n\pi) - 1)}{n^2\pi^2}cos\left(\frac{n\pi x}{2}\right)$$

$$= 1 - \frac{8}{\pi^2}\left(\frac{cos\pi x}{2} + \frac{1}{3^2}\frac{cos3\pi x}{2} + \frac{1}{5^2}\frac{cos5\pi x}{2} + \cdots\right)$$

The following Figure shows the periodic even extension of $f(x)$, $0 < x < 2$. We extend the function f from $0\ to\ -2$ by reflecting it in the vertical axis (y-axis), so that $f(-x) =$

$f(x)$, the original function together with its extension is even. Hence the Fourier expansion of the periodic continuation g will contain only cosine terms (including, of course, $\frac{a_0}{2}$).

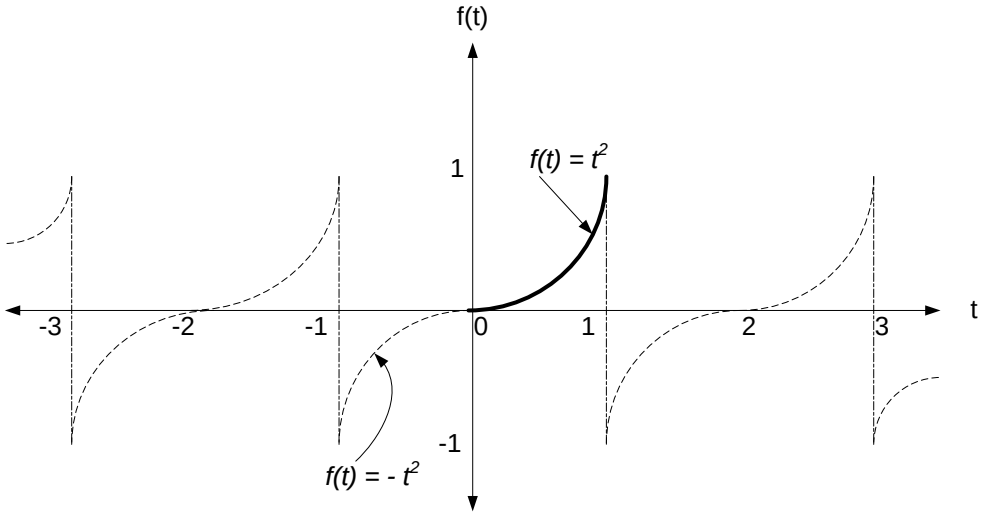


Peroidic even extension of f(x)

Example 1.3.4: What is the half-range sine expansion of the function

$$f(t) = t^2 \qquad 0 \le t < 1$$

Solution: We know that it **cannot** be obtained *FS* contains only sines **unless in the case of odd functions**. Therefore, we have to find *FS* of the periodic odd extension $h(t)$ of $f(t)$ shown in the following Figure.



Periodic odd extension of f(t)

Here, $L = 1$, since $h(x)$ is an odd function , then $a_n = 0$. Also,

$$b_n = \frac{2}{L}\int_0^L h(t) sin\frac{n\pi t}{L}\,dt = \frac{2}{1}\int_0^1 t^2\, sin\left(\frac{n\pi t}{1}\right)dt$$

Using the method of integration by parts, we can find the integral,

$$b_n = 2\left[\frac{-t^2\cos n\pi t}{n\pi} + \frac{2t\sin n\pi t}{n^2\pi^2} + \frac{2\cos n\pi t}{n^3\pi^3}\right]_0^1$$

$$b_n = 2\left[\frac{-1\cos n\pi}{n\pi} + \frac{2\sin n\pi}{n^2\pi^2} + \frac{2\cos n\pi}{n^3\pi^3}\right] - 2\left[\frac{2}{n^3\pi^3}\right]$$

$$b_n = \frac{-2n^2\pi^2\cos n\pi + 4n\pi\sin n\pi + 4cosn\pi}{n^3\pi^3} - \frac{4}{n^3\pi^3}$$

$$b_n = \frac{-2cosn\pi(n^2\pi^2 - 2) + 4(n\pi\sin n\pi - 1)}{n^3\pi^3}$$

$$b_n = \frac{-2(n^2\pi^2 - 2) - 4}{n^3\pi^3} = \frac{-2}{n\pi} + \frac{4}{n^3\pi^3} - \frac{4}{n^3\pi^3} = \frac{-2}{n\pi} \quad ; \quad n \ \ even$$

$$b_n = \frac{2(n^2\pi^2 - 2) - 4}{n^3\pi^3} = \frac{2}{n\pi} - \frac{4}{n^3\pi^3} - \frac{4}{n^3\pi^3} = \frac{2}{n\pi} - \frac{8}{n^3\pi^3} \quad ; \quad n \ \ odd$$

Then,

$$b_n = \begin{cases} \dfrac{-2}{n\pi} & for\ \ n\ \ even \\ \dfrac{2}{n\pi} - \dfrac{8}{n^3\pi^3} & for\ \ n\ \ odd \end{cases}$$

Then,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{1}\right) \quad ; \ \ where\ b_n\ is\ given\ above$$

Example 1.3.5: Expand the function $f(x) = \cos x$ in terms of *F.S.* consists of only sines on $\left(0, \frac{\pi}{2}\right)$.

Solution: We know that it **cannot** be obtained *F.S.* contains only sines unless in the case of ***odd*** functions. Therefore, we have to find *F.S.* of the periodic ***odd extension*** $h(x)$ of $f(x)$. Fourier series of $h(x)$ converges to $h(x)$ at each point of continuity of $h(x)$ since $h(x) = f(x)$ on $\left(0, \frac{\pi}{2}\right)$. That is, *F.S.* of $h(x)$ will converge to $f(x) = \sin x$ on $\left(0, \frac{\pi}{2}\right)$.

Here, $L = \frac{\pi}{2}$, and $a_n = 0$ since $h(x)$ is an ***odd*** function. Also,

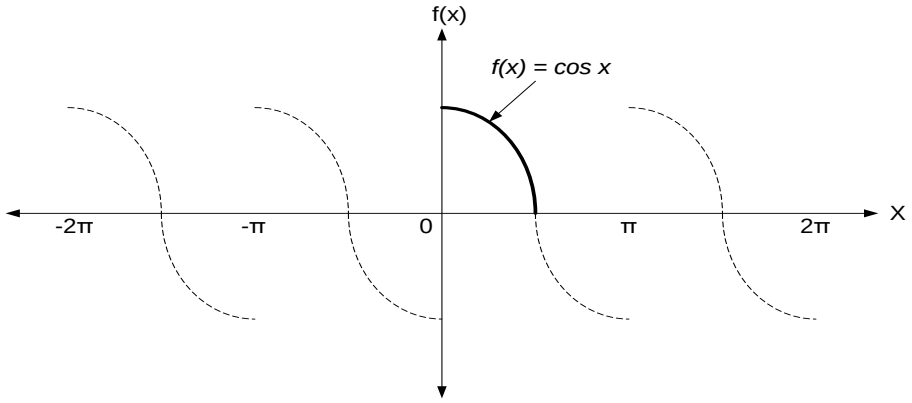
$$b_n = \frac{2}{L}\int_0^L h(x) sin\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} cos(x)\, sin(2nx)\,dx$$

$$b_n = \frac{-2}{\pi}\left[\frac{cos(2n+1)\,x}{2n+1} + \frac{cos(2n-1)\,x}{2n-1}\right]\Bigg|_0^{\frac{\pi}{2}} = \frac{8n}{\pi(4n^2-1)}$$

Then,

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin(2nx).$$

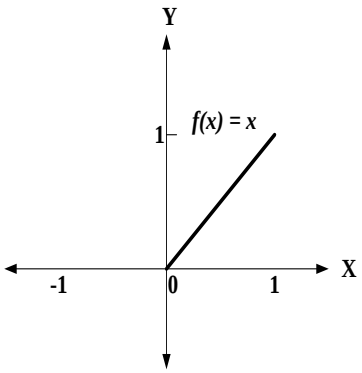
The periodic odd extension $h(x)$ of $f(x)$ shown in the following Figure.



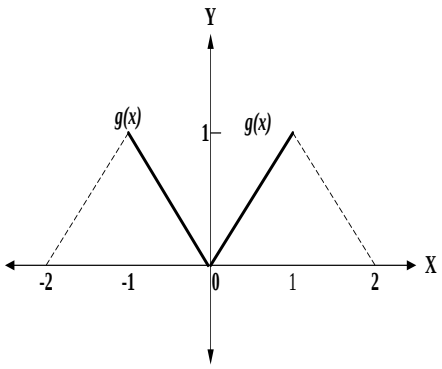
Periodic odd extension of f(x) 0 < x < π / 2

Example 1.3.6: Find the half-range expansion of the function $f(x) = x$ for $0 < x < 1$.

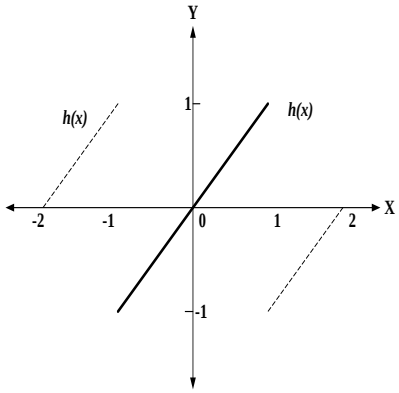
Solution: The graph of the even and odd extensions is shown in the following Figure.



(a) $f(x) = x \quad 0 < x < 1$



(b) *Even extension of f, period 2*



(c) *Odd extension of f, period 2*

For the even extension, we have

$$x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x \quad for\ all\ \ 0 \le x \le 1. \qquad (1)$$

Because f is continuous and piecewise smooth, then the Fourier series converges to $f(x)$ for all x. It is a remarkable fact that the cosine series and the sine series have the same values on the intervals $(0,1), (2,3), (-2,-1), \ldots$

For the odd extension, we have

$$b_n = \frac{2}{1}\int_0^1 x \sin n\pi x \ dx = \frac{2(-1)^{n+1}}{n\pi}.$$

Hence

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x \quad for\ all\ \ 0 \le x < 1.$$

Exercises A:

(1) If $f(x)$ is an ***even*** periodic function which satisfies the Dirichlet conditions, prove that the coefficients in the Fourier series of $f(x)$ are given by the formulas

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad , \quad b_n = 0$$

where $2L$ is the period of $f(x)$.

(2) If $f(x)$ is an ***odd*** periodic function which satisfies the Dirichlet conditions, prove that the coefficients in the Fourier series of $f(x)$ are given by the formulas

$$a_n = 0 \quad , \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

where $2L$ is the period of $f(x)$.

(3) The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function.

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & -\pi \le x < 0 \end{cases}$$

The periodic expansion of this function is called the **square wave function**. More generally, the input to an electrical circuit that switches from a high to a low state with time period T can be represented by the general square wave function with the following formula

$$f(x) = \begin{cases} 1 & 0 \le x < \dfrac{T}{2} \\ -1 & -\dfrac{T}{2} \le x < 0 \end{cases}$$

Find the Fourier series of the square wave and the general square wave.

Ans: For the square wave is

$$f(x) = \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}sin(nx) = \frac{4}{\pi}\sum_{k=0}^{\infty}sin\frac{(2k+1)x}{(2k+1)} = \frac{4}{\pi}\left(sin\,x + \frac{sin3x}{3} + \frac{sin5x}{5} + \ldots\right)$$

For the general square wave is

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}sin\frac{2(2k+1)\pi x}{T}$$

(4) Consider the square wave

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & -\pi \le x < 0 \end{cases}$$

and $f(x) = f(x + 2\pi)$. Find the Fourier series of the square wave.

Ans: $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(sinx + \frac{sin3x}{3} + \frac{sin5x}{5} + \cdots\right)$

Exercises B:

(1) Sketch and expand the function $f(x) = x+1$ in terms of *FS* consists of only sines on $(0,2)$.

(2) Sketch and expand the function $f(t) = t - t^2$ in terms of *FS* consists of only cosines on $(0,1)$.

(3) Sketch and expand the function $f(t) = t - t^2$ in terms of *FS* consists of only sines on $(0,1)$.

(4) Find sine series of $f(x)$ on $(0,2)$, where

$$f(x) = \begin{cases} x & x \leq 1 \\ 2 & x > 1 \end{cases}$$

(5) Find cosine series of $f(x)$ on $(0,2)$, where

$$f(x) = \begin{cases} x & x < 1 \\ 2 & x \geq 1 \end{cases}$$

(6) Find sine series of $f(x) = 1$ on $(0,5)$.

(7) Find cosine series of $f(x) = x$ on $(0,3)$.

(8) Find sine series of $f(x) = e^x$ on $(0,\pi)$.

(9) Find cosine series of $f(x) = e^x$ on $(0,\pi)$.

(10) Find cosine series of $f(x)$ on $(0,3)$, where

$$f(x) = \begin{cases} 0 & x \leq 2 \\ 2 & x > 2 \end{cases}$$

Answers:

1. $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-3\cos(n\pi)}{n} sin\frac{n\pi x}{2}$

2. $f(t) = \frac{1}{6} - \frac{4}{\pi^2}\left(\frac{cos2\pi t}{4} + \frac{cos4\pi t}{16} + \frac{cos6\pi t}{36} + \frac{cos8\pi t}{64} + \cdots\right)$

3. $f(t) = \frac{8}{\pi^3}\left(\frac{sin\pi t}{1} + \frac{sin3\pi t}{27} + \frac{sin5\pi t}{125} + \frac{sin7\pi t}{343} + \cdots\right)$

4. $\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2} sin\frac{n\pi}{2} + \frac{2}{n\pi} cos\frac{n\pi}{2} - \frac{4}{n\pi}\cos(n\pi)\right)\frac{\sin(n\pi x)}{2}$

5. $\frac{5}{4} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2} cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{2}{n\pi} sin\frac{n\pi}{2}\right) cos\frac{n\pi x}{2}$

6. $1 = \sum_{n=1}^{\infty}\frac{2}{n\pi}[1-(-1)^n] sin\frac{n\pi x}{5} = \frac{4}{\pi}\left(sin\frac{\pi x}{5} + \frac{1}{3} sin\frac{3\pi x}{5} + \frac{1}{5} sin\frac{5\pi x}{5} + \cdots\right)$

7. $x = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{6}{n^2\pi^2}[(-1)^n - 1] cos\frac{n\pi x}{3} = \frac{3}{2} - \frac{12}{\pi^2}\left(cos\frac{\pi x}{3} + \frac{1}{9} cos\frac{3\pi x}{3} + \frac{1}{25} cos\frac{5\pi x}{3} + \cdots\right)$

8. $e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+n^2}[1 - e^{\pi}(-1)^n]\sin(nx)$

9. $e^x = \frac{1}{\pi}(e^{\pi} - 1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+n^2}[e^{\pi}(-1)^n - 1]\cos(nx)$

10. $f(x) = \frac{4}{\pi}\left(\frac{1}{2} sin\frac{\pi x}{3} - \frac{3}{4} sin\frac{2\pi x}{3} + \frac{2}{3} sin\frac{3\pi x}{3} + \cdots\right) where, cos\frac{6\pi}{3} = 1, cos\frac{4\pi}{3} = -\frac{1}{2}, cos\frac{2\pi}{3} = -\frac{1}{2}$

Remark: So, at this point we have the following:

The full range Fourier series for $f(x)$ in the range $[-L, L]$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi x}{L}\right)dx, and\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\left(\frac{n\pi x}{L}\right)dx \ldots . (2)$$

Note that the range 0 to 2*L* (or, alternatively, –*L* to *L*) is called the *full range* because it contains a full *wavelength* of the periodic function. Notice also that both the ranges 0 to 2*L* and –*L* to *L* are exactly *equivalent*.

In the range −*L* to *L* neither sine nor cosine waves form a complete set. You remember that if a function defined between −*L* and *L* has *odd symmetry*, it can be represent as a *sine*

series. If a function has *even symmetry*, it can be represented as a cosine series. However, in the general case, to represent a function of arbitrary symmetry, we need to include both sine and cosine terms in the representation.

1.4 The Complex Form of Fourier Series

It is more convenient to express F.S., instead of (Eqs.1&2) in the above remark, in complex form using the complex exponentials; e^{inx}, $n \in \mathbb{Z}$, by substituting the exponential equivalents of the cosine and sine terms into the original form of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}}{2} + b_n \frac{e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}}{2i} \right)$$

due to the simple computational properties of these functions. Remember that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad and \ \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

The complex exponential has the following definition.

Definition: For any real number x, the complex exponential "Euler's Formula" is

$$e^{ix} = \cos x + i \sin x \quad where \ \ i = \sqrt{-1}\,.$$

Lemma: For any $x, y \in \mathbb{R}$

1. $e^{i(x+2\pi)} = e^{ix}$
2. $\left|e^{ix}\right| = 1$
3. $\overline{e^{ix}} = e^{-ix}$
4. $e^{ix} e^{iy} = e^{i(x+y)}$
5. $\frac{e^{ix}}{e^{iy}} = e^{i(x-y)}$
6. $\frac{d}{dt}\{e^{ix}\} = ie^{ix}$

Thmeorem: The set of functions $e_n(t) = \left\{\frac{e^{int}}{\sqrt{2\pi}}, \ \ n = \cdots, -2, -1, 0, 1, 2, \ldots\right\}$ is orthonormal in the Hilbert space $L^2[(-\pi.\pi)]$ of square integrable functions on $(-\pi, \pi)$.

Proof: We must show $\langle e^{int} e^{imt} \rangle_{L^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{imt}}\ dt == \begin{cases} 1, & if\ m = n \\ 0, & if\ m \neq n \end{cases}$

Using the third, fourth, and sixth properties stated in above lemma, we have

$$\int_{-\pi}^{\pi} e^{int} \overline{e^{imt}}\ dt = \int_{-\pi}^{\pi} e^{int} e^{-imt} dt = \int_{-\pi}^{\pi} e^{i(n-m)t} dt = \frac{e^{i(n-m)t}}{i(n-m)}\Big|_{-\pi}^{\pi} = 0\ ; if\ n \neq m$$

If $m = n$; $e^{int} e^{-imt} = e^{int} e^{-int} = e^0 = 1$, and so, $\langle e^{int} e^{imt} \rangle_{L^2} = 2\pi$.

Note: We can use (and need) this orthonormality to find the coefficients of the following complex F.S.

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}\ ; \ldots\ldots\ldots\ldots\ldots (3)$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{-in\pi x}{L}} dx, \ldots\ldots\ldots\ldots (4)$$

Now, we are going to make use of the well celebrated Euler's Formula so that we can rewrite (Eqs.1&2) in its complex form (Eqs.3&4).

Derivation of complex series results:

Using complex exponentials via De Moiver's theorem $e^{i\theta} = \cos\theta + i \sin\theta$, so

$$cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}})$$

$$sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i}(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}})$$

and our F.S. becomes

$$\frac{f(x_+) + f(x_-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2} \right) e^{\frac{in\pi x}{L}} + \left(\frac{a_n + ib_n}{2} \right) e^{\frac{-in\pi x}{L}} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{L}}$$

where,

$$c_0 = \frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx - i \frac{1}{2L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, n > 0 ;$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{\frac{-in\pi x}{L}} dx \,, n > 0$$

and

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}} dx \,, n > 0$$

Example 1.4.1: Find the complex form of the Fourier series of the periodic function whose definition in one period is $f(t) = e^{-t}$, $-1 < t < 1$.

Solution: Since $2L = 2 \Rightarrow L = 1$, and hence we have

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(t)\, e^{\frac{-in\pi t}{L}} dt \,, n > 0$$

Then

$$c_n = \frac{1}{2} \int_{-1}^{1} e^{-t}\, e^{-in\pi t} dt = \frac{1}{2} \left[\frac{e^{-(1+ni\pi)t}}{-(1+ni\pi)} \right]_{-1}^{1} = \frac{e^{-(1+ni\pi)} - e^{(1+ni\pi)}}{-2(1+ni\pi)} = \frac{e e^{ni\pi} - e^{-1} e^{-ni\pi}}{2(1+ni\pi)}$$

Now,

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \Rightarrow e^{ni\pi} = e^{-ni\pi} = (-1)^n$$

Hence

$$c_n = \frac{(-1)^n}{(1+ni\pi)}\frac{e-e^{-1}}{2} = \frac{(-1)^n(1-ni\pi)\,sinh\,1}{(1+n^2\pi^2)}$$

The expansion of $f(t)$ is therefore

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-ni\pi)\,sinh\,1}{(1+n^2\pi^2)} e^{ni\pi t}$$

This, of course, can be converted into the real trigonometric form, for we have, by definition,

$$c_n = \frac{a_n - ib_n}{2} \quad and \quad c_{-n} = \frac{a_n + ib_n}{2}$$

and thus, by adding and subtracting these expressions, we obtain

$$a_n = c_n + c_{-n} \quad and \quad b_n = i(c_n - c_{-n})$$

Therefore, in this problem

$$a_n = \frac{(-1)^n(1-ni\pi)\,sinh\,1}{(1+n^2\pi^2)} + \frac{(-1)^n(1+ni\pi)\,sinh\,1}{(1+n^2\pi^2)} = \frac{(-1)^n.2.sinh\,1}{1+n^2\pi^2}$$

$$b_n = i\left[\frac{(-1)^n(1-ni\pi)\,sinh\,1}{(1+n^2\pi^2)} - \frac{(-1)^n(1+ni\pi)\,sinh\,1}{(1+n^2\pi^2)}\right] = \frac{(-1)^n.2n\pi.sinh\,1}{1+n^2\pi^2}$$

Hence we can also write

$$f(t) = \sinh 1 - 2\sinh 1\left(\frac{\cos\pi t}{1+\pi^2} - \frac{\cos 2\pi t}{1+4\pi^2} + \frac{\cos 3\pi t}{1+9\pi^2} - \cdots\right) -$$

$$2\pi\sinh 1\left(\frac{\sin\pi t}{1+\pi^2} - \frac{2\sin 2\pi t}{1+4\pi^2} + \frac{3\sin 3\pi t}{1+9\pi^2} + \cdots\right)$$

Example 1.4.2: We revisit our ***Example 1.2.1*** of expanding $f(x) = x^2$ on $[-L, L]$ (we could choose $L = \pi$ if we wished) to show **the complex Fourier** approach in action, .The

general expression for the Fourier coefficients, c_n, takes one of the following forms; depending on whether or not *n* is zero:

$$c_{n=0} = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{L^2}{3} \;;\; c_{n\neq 0} = \frac{1}{2L}\int_{-L}^{L} x^2 e^{\frac{-in\pi x}{L}} dx = \frac{2L^2(-1)^n}{n^2\pi^2}$$

See below for details of how to do the second integral. We notice that in this case all the c_n are real, but this is not the case in general.

Now, we want to calculate $c_{n\neq 0} = \frac{1}{2L}\int_{-L}^{L} x^2 e^{\frac{-in\pi x}{L}} dx$?

We should change variables to make the exponent more simple (whilst keeping *y* real) i.e. set $y = \frac{n\pi x}{L}$, for which $dy = \left(\frac{n\pi}{L}\right)dx$. The integration limits become $\pm n\pi$:

$$c_n = \frac{1}{2L}\int_{-n\pi}^{n\pi}\left(\frac{L}{n\pi}\right)\left(\frac{L^2y^2}{n^2\pi^2}\right)e^{-iy}dy = \frac{L^2}{n^3\pi^3}\int_{-n\pi}^{n\pi} y^2 e^{-iy}dy\,.$$

Now we want to integrate by parts:

Let $u = y^2 \rightarrow du = 2ydy \; and \; \frac{dv}{dy} = e^{-iy} \rightarrow v = \frac{e^{-iy}}{-i} = ie^{-iy}$. So,

$$c_n = \frac{L^2}{2n^3\pi^3}\left\{\left[iy^2e^{-iy}\right]_{-n\pi}^{n\pi} - \int_{-n\pi}^{n\pi} 2yie^{-iy}dy\right\} = \frac{L^2}{2n^3\pi^3}\left\{\left[iy^2e^{-iy}\right]_{-n\pi}^{n\pi} - 2i\int_{-n\pi}^{n\pi} ye^{-iy}dy\right\}$$

Now, let $u = y \rightarrow \frac{du}{dy} = 1$. So,

$$c_n = \frac{L^2}{2n^3\pi^3}\left\{\left[iy^2e^{-iy}\right]_{-n\pi}^{n\pi} - 2i(iye^{-iy})_{-n\pi}^{n\pi} - \int_{-n\pi}^{n\pi} ie^{-iy}dy\right\}$$

$$c_n = \frac{L^2}{2n^3\pi^3}\left\{\left[iy^2e^{-iy}\right]_{-n\pi}^{n\pi} - 2i(iye^{-iy})_{-n\pi}^{n\pi} - i(ie^{-iy})_{-n\pi}^{n\pi}\right\}$$

$$c_n = \frac{L^2}{2n^3\pi^3}(iy^2e^{-iy} - 2i.i.ye^{-iy} + 2i.i.ie^{-iy})_{-n\pi}^{n\pi}$$

$$c_n = \frac{L^2}{2n^3\pi^3}[e^{-iy}(iy^2 + 2y - 2i)]_{-n\pi}^{n\pi}$$

We can now just substitute the limits in, using $e^{-ni\pi} = e^{ni\pi} = (-1)^n$, so (e^{-iy} has the same value at both limits). Thus,

$$c_n = \frac{L^2}{2n^3\pi^3}[e^{-iy}2y]_{-n\pi}^{n\pi} = \frac{L^2}{2n^3\pi^3}[2n\pi e^{-in\pi} - (-2n\pi)e^{in\pi}]$$

$$c_n = \frac{L^2}{2n^3\pi^3} \times 4n\pi(-1)^n = \frac{2L^2(-1)^n}{n^2\pi^2}.$$

Exercise 1.4.2: Find the complex F.S. of the function $f(t)$, where f is defined as,

$$f(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & -\pi \le t < 0 \end{cases}.$$

Ans: $\sum_{n=-\infty}^{\infty} c_n\, e^{int} = \sum_{k=-\infty}^{\infty} \frac{-2i}{(2k+1)\pi} e^{i(2k+1)t}$. $c_n = \frac{i[(1-\cos n\pi)]}{n\pi} = \begin{cases} -\frac{2i}{n\pi} & n\ is\ odd \\ 0 & n\ is\ even \end{cases}$.

Fourier Serie and Series Expansions

We can sometimes exploit Fourier series to either give us series approximations for numerical quantities, or to give us the result of summing a series.

Example1.4.3: Consider $f(x) = x^2$ on $[-L, L]$, which we expanded as a Fourier series. This function is an even function, so we know $b_n = 0$. The other coefficients are:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2 dx = \frac{2}{L}\int_{0}^{L} x^2 dx = \frac{2L^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2L^2}{n^3\pi^3}[y^2 siny + 2ycosy - 2siny]_0^{n\pi}$$

$$= \frac{2L^2}{n^3\pi^3} \times 2n\pi(-1)^n = \frac{4L^2(-1)^n}{n^2\pi^2}.$$

Thus, our F.S. is

$$S(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right), \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Inside the expansion range $S(x)$ agrees exactly with the original function $f(x)$. Outside, however, it ***does not***: $f(x)$ keeps rising quadratically, whereas $S(x)$ repeats with period $2L$. We say the Fourier series has periodically extended the function $f(x)$ outside the expansion range.

Now, let's choose the expansion range to be $-\pi \to \pi$ (i.e. we'll set $L = \pi$). At $x = 0$, we have $f(x) = S(x) = 0$. Substituting into Eq.(1), we have

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \to \frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \ldots\ldots\ldots\ldots\ldots (2)$$

This result can be useful in two ways:

1. We solve a physics problem, and find the answer as a sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. Using the above result we can replace the sum by $\frac{\pi^2}{12}$.

2. We need a numerical approximation for π. We can get this by truncating the sum at some upper value *n* = *N* and adding together all the terms in the sum.

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots, \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let's call this approximation π_N ;

$$\pi_N = \sqrt{12 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

You can see how π_N approach π as we increase *N*. We can get different series approximations by considering different values of *x* in the same Fourier series expansions. For instance, consider *x* = *π*. This gives:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n \to \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}, \ldots\ldots\ldots\ldots\ldots (5)$$

Fourier series (real or complex) are very good ways of approximating functions in a finite range, by which we mean that we can get a good approximation to the function by using only the first few modes (i.e. truncating the sum over *n* after some low value *n* = *N*).

This is how ***music compression*** works in MP3 players, or how ***digital images are compressed*** in JPEG (***Joint Photographic Experts Group***) form: we can get a ***good approximation*** to the true waveform by using only a limited number of modes, and so all modes below a certain amplitude are simply ignored. We saw a related example of this in our approximation to π using (Eq.4).

Moreover, we saw that the conditions imposed on $f(x)$ in the Theorem regarding Dirichlet's criterion (***page 16***), are ***sufficient but not necessary***. This means that if the conditions are satisfied, the convergence is guaranteed. However, if they are not satisfied, the series ***may or may not converge***.

How close does it get? Convergence of Fourier expansions

We have seen that the Fourier components generally get smaller as the mode number *n* increases. If we truncate the Fourier series after *N* terms, we can define an error D_N that measures how much the truncated Fourier series differs from the original function: i.e. if

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right), \dots\dots\dots\dots\dots\dots (6)$$

We define the error as

$$D_N = \int_{-L}^{L} |f(x) - f_N(x)|^2 dx \geq 0, \dots\dots\dots\dots\dots\dots\dots\dots (7)$$

That is, we square the difference between the original function and the truncated Fourier series at each point *x*, then integrate across the full range of validity of the Fourier series. Technically, this is what is known as an L^2 norm.

Some things you should know, but which we will not prove: if *f* is reasonably well-behaved (no non-integrable singularities, and only a finite number of discontinuities), the Fourier series is optimal in the least-squares sense - i.e. *if we ask what Fourier coefficients will minimise D_N for some given N*, they are exactly the coefficients that we obtain by solving the full Fourier problem.

Furthermore, as $N \to \infty$, $D_N \to 0$. This sounds like we are guaranteed that the Fourier series will represent the function exactly in the limit of infinitely many terms. But looking at the

equation for D_N, it can be seen that this is not so: it's always possible to have (say) $f_N = 2f$ over some range Δx, and the best we can say is that Δx must tend to zero as N increases.

As an **example** of how Fourier series converge (or not), consider the signum function which picks out the sign of a variable:

$$f(x) = signum(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}, \dots\dots\dots\dots\dots\dots\dots (8)$$

The following Table shows the Error D_N on the N-term truncated Fourier series approximation to the signum function (Eq.8):

N	D_N
10	0.0808
50	0.0162
100	0.0061
250	0.0032

which we will expand in the range $-1 \leq x \leq 1$ (i.e. we set $L = 1$). The function is odd, so $a_n = 0$ and we find

$$b_n = 2\int_0^1 sin\,(n\pi x)dx = \frac{2}{n\pi}[1 - (-1)^n], \dots\dots\dots\dots\dots\dots\dots (9)$$

$f(x)$ has discontinuities at $x = 0$ and $x = \pm L = \pm 1$ (due to the periodic extension). In the above Table, we show the error D_N for the signum function for increasing values of D_N. As expected the error decreases as N gets larger, but relatively slowly. We'll see why this is in the next section.

Ringing artefacts and the Gibbs phenomenon

We saw above that we can define an error associated with the use of a truncated Fourier series of N terms to describe a function. Note that D_N measures **the total error** by integrating the deviation at each value of x over the full range. It does not tell us whether the deviations between $f_N(x)$ and $f(x)$ were large and concentrated at certain values of x, or smaller and more evenly distributed over all the full range. An interesting case is when we try to describe a function with a finite discontinuity (i.e. jump) using a truncated Fourier series, such as our discussion of the signum function above.

In the following, **Fig.1**, we plot the original function $f(x)$ and the truncated Fourier series for various N. We find that the truncated sum works well, except near the discontinuity.

Here the function overshoots the true value and then has a **"damped oscillation"**. As we increase *N* the oscillating region gets smaller, but the overshoot remains roughly the same size (about 18%).

This overshoot is known as the ***Gibbs phenomenon***. Looking at the plot, we can see that it tends to be associated with extended oscillations either side of the step, known as ***"ringing artefacts"***. Such artifacts will tend to exist whenever we try to describe sharp transitions with Fourier methods, and are one of the reasons that MP3s can sound bad when the compression uses too few modes. We can reduce the effect by using a smoother method of Fourier series summation.

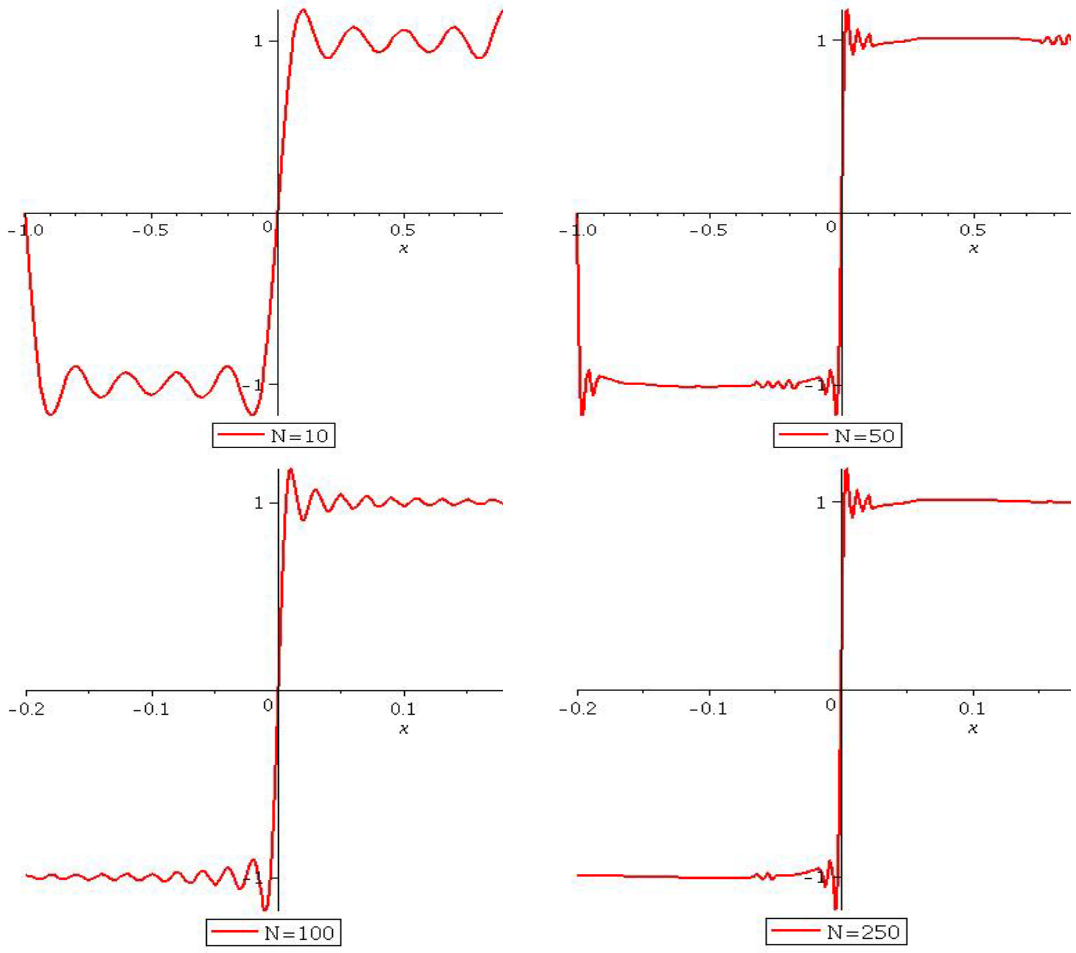


Figure1: The Gibbs phenomenon for truncated Fourier approximations to the signum function (Eq.8). Note the different *x*-range in the lower two panels.

Note: From our study of Hilbert spaces we already know that Bessel's inequality holds:

$$(f,f) \geq \sum_{-\infty}^{\infty} |(f,e_n)|^2\, dt, \quad or$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt \geq \sum_{n=-\infty}^{\infty} |c_n|^2, \ \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Lemma (Bessel's inequality): Suppose that f is integrable on the interval [-L,L]. Let a_n, b_n, and c_n be the Fourier coefficients of f, then

$$\frac{1}{L}\int_{-L}^{L} |f(t)|^2\, dt \geq \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) = 2\sum_{n=-\infty}^{\infty} |c_n|^2.$$

Proof: Let S_N denotes the partial sums $S_N = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n cos\,(\frac{n\pi x}{L}) + b_n sin\,(\frac{n\pi x}{L}\right)$. Now,

$$[f(x) - S_N(x)]^2 = f^2(x) - 2f(x)S_N(x) + S_N^2(x)$$

Consider the least-square error defined to be

$$\frac{1}{L}\int_{-L}^{L} [f(x) - S_N(x)]^2 dx = \frac{1}{L}\left\{\int_{-L}^{L} f^2(x)dx - 2\int_{-L}^{L} f(x)S_N(x)dx + \int_{-L}^{L} S_N^2(x)dx\right\}$$

$$= \frac{1}{L}\{\langle f,f\rangle - 2\langle f,S_N\rangle + \langle S_N,S_N\rangle\}; \quad Now,$$

$$\langle S_N,S_N\rangle = \int_{-L}^{L}\left[\frac{a_0}{2} + \sum_{n=1}^{N} a_n\, cos\left(\frac{n\pi x}{L}\right) + b_n sin\,(\frac{n\pi x}{L})\right]^2 dx$$

$$\langle S_N,S_N\rangle = \frac{a_0^2}{2}L + \sum_{n=1}^{N} a_n^2 \int_{-L}^{L} cos^2\left(\frac{n\pi x}{L}\right)dx + b_n^2 \int_{-L}^{L} sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$= L\left[\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right]$$

In addition,

$$\langle f, S_N \rangle = \int_{-L}^{L} f(x) S_N(x) dx$$

$$= \frac{a_0}{2} \int_{-L}^{L} f(x) dx + \sum_{n=1}^{N} a_n \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx + b_n \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \left[\frac{a_0^2}{2} L + \sum_{n=1}^{N} a_n^2 L + b_n^2 L \right]$$

Therefore,

$$\frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 dx = \frac{1}{L} \langle f, f \rangle - \left\{ \frac{a_0^2}{2} \sum_{n=1}^{N} a_n^2 + b_n^2 \right\}; \ \ and \ since,$$

$D_N[f, S_N] = \int_{-L}^{L} |f(x) - f_N(x)|^2 dx \geq 0$, it follows that

$$\frac{a_0^2}{2} \sum_{n=1}^{N} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} f^2(x) dx = \frac{1}{L} \langle f, f \rangle = E[f]$$

where $E[f]$ is known as the **energy** of the $2L$-periodic function f. Using the relations between a_n, b_n, and c_n, we complete the proof.

Note: In the above Lemma, if we replace [-L,L] by [0,T]; we rewrite Bessel's inequality as:

Suppose that f is integrable on the interval [0,T]. Let a_n, b_n, and c_n be the Fourier coefficients of f. Assume that $f(x)$ has the period T. then

$$\frac{2}{T} \int_{0}^{T} |f(t)|^2 \, dt \geq \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) = 2 \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Proof: Since $f(x)$ has the period T, the F.S. corresponding to $f(x)$ (with $\omega = \frac{2\pi}{T}$) is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n cos\,(n\omega x) + b_n sin\,(n\omega x))$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_0^T f(x)\,cos(n\omega x)\,dx\;; n = 0,1,2,\ldots \; and \;\; b_n = \frac{2}{T}\int_0^T f(x)\,sin(n\omega x)\,dx\,; n = 1,2,..$$

Let S_N denotes the partial sums $S_N = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)\right.$. Now,

$$\int [f(t) - S_N(t)]^2\,dt = \int f^2(t)dt - 2\int f(t)S_N(t)dt + \int S_N^2\,(T)dt$$

From definition of the Fourier coefficients, it follows that

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) = \frac{2}{T}\int_0^T f(t)S_N(t)dt$$

By multiplying out the terms of S_N^2 and taking into account the orthogonality relations of the trigonometric functions, it follows that,

$$\frac{2}{T}\int_0^T {S_N}^2(t)dt = \frac{2}{T}\int_0^T f(t)S_N(t)dt\,.\; Therefore$$

$0 \le \frac{2}{T}\int_0^T [f(t) - S_N(t)]^2\,dt = \frac{2}{T}\int_0^T f^2(t)\,dt - \left\{\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right\}$. Since f is itegrable, we may let *N* tend to infinity

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \le \int_0^T f^2(t)dt < \infty\,.$$

An immediate consequence is the Riemann-Lebesgue Lemma.

Thmeorem: (Riemann-Lebesgue Lemma). Suppose f is a piecewise continuous function on the interval [a,b], then

$$\lim_{n\to\infty} \int_a^b f(x)\, cos(nx)\, dx = \lim_{n\to\infty} \int_a^b f(x)\, sin(nx)\, dx = 0.$$

By definition, a piecewise continuous function has only a finite number of discontinuities. This hypothesis can be weakened considerably.

Proof: The intuitive idea behind the proof is that as *n* gets large, sin(*nx*) and cos(*nx*) oscillate much more rapidly than does f . If *n* is large, $f(x)$ is nearly constant on two adjacent periods of sin(*nx*) or cos(*nx*).

Now, we give the following analytical proof of the theorem in the case of a differentiable function f. Consider the term

$$\int_a^b f(x)\, cos(nx)\, dx.$$

We integrate by parts with $u = f$ and *dv* = cos(*nx*) to obtain

$$\int_a^b f(x)\, cos(nx)\, dx = \frac{sin\,(nx)}{n} f(x)\Big|_a^b - \int_a^b \frac{sin\,(nx)}{n} f'(x)\, dx$$

$$= \frac{sin(nb)\, f(b) - sin(na)\, f(a)}{n} - \int_a^b \frac{sin\,(nx)}{n} f'(x)\, dx$$

As *n* gets large in the denominator, the expression on the right converges to zero.

Lemma (Riemann-Lebesgue, **weak form**). $lim_{|n|\to\infty} \int_0^{2\pi} f(t) e^{-int} dt = 0.$

Thus, as $|\, n\,|$ gets large the Fourier coefficients go to zero. If *f* is a real-valued function then $\overline{c_n} = c_{-n}$ for all *n*. If we set

$$c_n = \frac{a_n - ib_n}{2}, \qquad n = 0,1,2, \ldots$$

$$c_{-n} = \frac{a_n + ib_n}{2}, \qquad n = 1,2,3, \ldots$$

and rearrange terms, we get the real version of Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\, dt, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\, dt,$$

with Bessel inequality

$$\frac{1}{\pi}\int_0^{2\pi} |f(t)|^2\, dt \geq \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

and Riemann-Lebesgue Lemma

$$\lim_{n\to\infty}\int_0^{2\pi} f(t)\cos(nt)\, dt = \lim_{n\to\infty}\int_0^{2\pi} f(t)\sin(nt)\, dt = 0$$

Remark: Again, one important consequence of the above Theorem is that only a finite number of Fourier coefficients are larger (in absolute value) than any given positive number.

This fact is the basic idea underlying the process of ***Data Compression***. One way to compress a signal is first to express it as a F.S., then discard all the small Fourier coefficients and retain (or transmit) only the finite number of Fourier coefficients that are larger than some given threshold.

1.5 Parseval's Theorem for Fourier Series

The integral of a squared periodic function (or squared modulus for complex functions) is often of interest in applications, e.g. representing the **energy** of a periodic signal

$$E = \int_0^{2L} |f(x)|^2 dx = (f, f)$$

Substituting the complex form of the Fourier series

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

and using the following orthogonality property of the complex exponentials

$$\left(e^{\frac{in\pi x}{L}}, e^{\frac{im\pi x}{L}}\right) = \int_0^{2L} e^{\frac{in\pi x}{L}} e^{\frac{-im\pi x}{L}} dx = 2L\delta_{mn}, m, n \in \mathbb{Z}$$

we immediately get

$$\int_0^{2L} |f(x)|^2 dx = 2L \sum_{-\infty}^{\infty} |c_n|^2, \qquad (complex\ version)\ \ldots\ldots\ldots (P)$$

This result is called ***Parseval's theorem***.

Using the following relations,

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, n > 0\ ; and\ c_{-n} = \frac{a_n + ib_n}{2}, n > 0$$

We can express Eq.(P) in terms of the a_n and b_n as

$$\int_0^{2L} |f(x)|^2 dx = L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right], \qquad (real\ version) \ldots\ldots (Q)$$

If $f \in L^2[0,2\pi]$, Eq.(P) and Eq.(Q) can be written respectively as

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2, \qquad (complex\ version) \ldots\ldots\ldots\ldots\ldots\ldots (R)$$

$$\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx = \left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right], \qquad (real\ version) \ldots\ldots\ldots\ldots\ldots (S)$$

Proving Parseval's theorem is easier in the complex case, so we will stick to this. First, let

$\emptyset_n(x) = e^{ik_n x} = e^{\frac{in\pi x}{L}}$; $k_n = \frac{n\pi}{L}$ is the wavenumber. $\emptyset_n^*(x) = \overline{\emptyset_n}(x) = e^{-ik_n x} = e^{\frac{-in\pi x}{L}}$;

Now, $f(x) = \sum_{n=-\infty}^{\infty} c_n \emptyset_n(x)$. We know that

$$|f(x)|^2 = f(x)f^*(x) = \sum_{-\infty}^{\infty} c_n \emptyset_n(x) \sum_{-\infty}^{\infty} c_m^* \emptyset_m^*(x)$$

We integrate over –L ≤ x ≤ L, noting the orthogonality of $\emptyset_n \; and \; \emptyset_m^*$:

$$\int_{-L}^{L} |f(x)|^2 \, dx = \sum_{m,n=-\infty}^{\infty} c_n c_m^* \int_{-L}^{L} \emptyset_n(x) \emptyset_m^*(x) dx = \sum_{m,n=-\infty}^{\infty} c_n c_m^* \, (2L\delta_{nm})$$

$$= 2L \sum_{m,n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

where we have used the orthogonality relation

$$\int_{-L}^{L} \emptyset_n(x) \emptyset_m^*(x) dx = \begin{cases} 2L & if \;\; m = n \\ 0 & otherwise \end{cases}$$

Remark:

(1) **One interpretation** of Parseval's formula is that the energy in a signal is the sum of the energies associated with the Fourier components.

(2) Parseval's formula can be **interpreted** as a kind of infinite dimensional version of Pythagoras' theorem (that the squared length of a vector is the sum of the squared components in any orthonormal basis). Indeed on [-*L*; *L*] the following functions form an orthonormal set (i.e. pairwise orthogonal and each having norm 1):

$$\begin{cases} c_0 = \dfrac{1}{\sqrt{2L}} \\ f_n(x) = \dfrac{1}{\sqrt{L}} sin \dfrac{n\pi x}{L}; n = 1,2,3, \ldots \\ g_n(x) = \dfrac{1}{\sqrt{L}} cos \dfrac{n\pi x}{L}; n = 1,2,3, \ldots \end{cases}$$

The Fourier series Eq.(*A*) with these slightly rescaled basic functions becomes

$$f(x) = \left(\sqrt{\frac{L}{2}} a_0 \right) c_0 + \sum_{n=1}^{\infty} \sqrt{L}\, a_n f_n(x) + \sqrt{L} b_n g_n(x)$$

and then Parseval's theorem Eq.(Q) is formally just Pythagoras' theorem in this infinite dimensional setting.

(3) If the series on the right side of Eq.(*R*) is truncated at some finite value *N*, then the right side can only get smaller resulting in the following inequality

$$\sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \|f\|^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (T)$$

This is known as ***Bessel's inequality*** (see Eq.10).

Example 1.5.1 : Write ***Parseval's inequality*** corresponding to the F.S. of the function f with period 8, where

$$f(x) = \begin{Bmatrix} -x & -4 \leq x \leq 0 \\ x & 0 \leq x \leq 4 \end{Bmatrix}$$

Solution: Note that 2*L* = 8, so *L* = 4. Now,

$$a_n = \frac{1}{4} \int_{-4}^{0} (-x) \cos\left(\frac{n\pi x}{4}\right) dx + \frac{1}{4} \int_{0}^{4} (x) \cos\left(\frac{n\pi x}{4}\right) dx = \frac{-8(1 - cos\,(n\pi))}{n^2\pi^2}; \;\; n \geq 1$$

$$a_0 = \frac{1}{4}\int_{-4}^{0}(-x)dx + \frac{1}{4}\int_{0}^{4}(x)dx = 4$$

$$b_n = \frac{1}{4}\int_{-4}^{0}(-x)\,sin\left(\frac{n\pi x}{4}\right)dx + \frac{1}{4}\int_{0}^{4}(x)\,sin\left(\frac{n\pi x}{4}\right)dx = 0 \ \ (\boldsymbol{check}!).$$

Therefore, the F.S. is,

$$2 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi)}{n^2}\cos\left(\frac{n\pi x}{4}\right). \quad (\boldsymbol{check}!!)$$

Now, Eq.(S) gives us the (real version) of ***Parseval's inequality***, thus,

$$\frac{1}{4}\int_{-4}^{4}f^2(x)dx = \frac{1}{4}\int_{-4}^{4}|x|^2\,dx = \frac{1}{4}\int_{-4}^{4}x^2dx = \frac{32}{3}; \quad and,$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{16}{2} + \sum_{n=1}^{\infty}\frac{64(1-cos\,(n\pi))^2}{n^4\pi^4} = 8 + \frac{64}{\pi^4}\left(\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \frac{4}{7^4} + \cdots\right)$$

Therefore, ***Parseval's inequality*** becomes

$$\frac{32}{3} = 8 + \frac{256}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right); which\ can\ be\ written\ as:$$

Then,

$$\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right) = \frac{\pi^4}{96}.$$

Example 1.5.2: Using the result of **Ex.1.5.1** above, compute $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

Solution: Let $S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right)$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) = \frac{\pi^4}{96} + \frac{S}{16}$$

Thus,

$$S = \frac{\pi^4}{90}.$$

Example 1.5.3: Consider again the sawtooth function $f(x) = x$ for $-L \le x \le L$. This function having period $2L$. It is an odd function, then $a_n = 0$ for all n.

$$b_n = \frac{2L}{n\pi}(-1)^{n+1},$$

$$f(x) = \frac{2L}{\pi}\left[sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}sin\left(\frac{3\pi x}{L}\right) + \cdots\right], \ldots\ldots(U)$$

If we substitute Eq.(*U*) into Parseval's formula Eq.(*Q*) we get

$$\int_{-L}^{L} x^2 dx = \frac{2L^3}{3} = L\sum_{m=1}^{\infty}\frac{4L^2}{m^2\pi^2}$$

giving the nice formula

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Example 1.5.4: Let the sawtooth function $f(x) = \frac{x^2}{2}$ defined on [-*L*,*L*]. show that the Fourier series representation of this function is

$$\frac{x^2}{2} = L^2\left[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2}cos\,(\frac{n\pi x}{L})\right], \ldots\ldots\ldots\ldots\ldots(V)$$

Then apply Parseval's formula to Eq.(*V*) to obtain

$$\sum_{m=1}^{\infty}\frac{1}{m^4} = \frac{\pi^4}{90}$$

Example 1.5.5: Let $f(t) = \begin{cases} 0, & t = 0 \\ \frac{\pi - t}{2}, & 0 < t < 2\pi \\ 0 & t = 2\pi \end{cases}$, and $f(t+2\pi) = f(t)$.

We have $a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - t}{2}dt$, and for $n \ge 1$, we have;

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - t}{2}cos(nt)\,dt = \frac{\frac{\pi - t}{2}sin\,(nt)}{nt}\Bigg|_0^{2\pi} + \frac{1}{2n\pi}\int_0^{2\pi} sin\,(nt)\,dt = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - t}{2} sin(nt)\, dt = -\frac{\frac{\pi - t}{2} cos\,(nt)}{nt}\Big|_0^{2\pi} - \frac{1}{2n\pi}\int_0^{2\pi} cos\,(nt)\, dt = \frac{1}{n},$$

Thus,

$$\frac{\pi - t}{2} = \sum_{n=1}^{\infty} \frac{sin\,(nt)}{n}, \qquad 0 < t < 2\pi.$$

By setting $t = \frac{\pi}{2}$ in this expansion we get an alternating series for $\frac{\pi}{4}$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$

Parseval's identity gives:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Example 1.5.6: $f(t) = \begin{cases} \frac{1}{2} & t = 0 \\ 1 & 0 < t < \pi \\ \frac{1}{2} & t = \pi \\ 0 & \pi < t < 2\pi \end{cases}$, and $f(t+2\pi) = f(t)$ (a step function).

We have $a_0 = \frac{1}{\pi}\int_0^{\pi} 1\, dt = 1$, and for $n \geq 1$, we have;

$$a_n = \frac{1}{\pi}\int_0^{\pi} 1\, cos(nt)\, dt = \frac{sin\,(nt)}{nt}\Big|_0^{\pi} = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 1\, sin(nt)\, dt = -\frac{cos\,(nt)}{nt}\Big|_0^{\pi} = \frac{(-1)^{n+1} + 1}{n\pi} = \begin{cases} \frac{2}{n\pi} & n\ odd \\ 0 & n\ even \end{cases},$$

Thus,

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{sin\,(2n-1)t}{(2n-1)}.$$

For $0 < t < \pi$, this gives:

$$\frac{\pi}{4} = sin\, t + \frac{sin\ 3t}{3} + \frac{sin\, 5t}{5} + \cdots$$

and for $\pi < t < 2\pi$, it gives

$$-\frac{\pi}{4} = sin\, t + \frac{sin\ 3t}{3} + \frac{sin\, 5t}{5} + \cdots$$

Parseval's equality becomes

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$$

Example 1.5.7: Consider once again the case of $f(x) = x^2$. The L.H.S. of Parseval's theorem is

$$\frac{1}{2L}\int_{-L}^{L} x^4\, dx = \frac{1}{5}L^4.$$

The complex coefficients were derived earlier, so the sum on the RHS of Parseval's theorem is

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = |c_0|^2 + \sum_{n\neq 0}^{\infty} |c_n|^2 = \left(\frac{L^2}{3}\right)^2 + 2\sum_{n=1}^{\infty}\left(\frac{2L^2(-1)^n}{n^2\pi^2}\right)^2 = \frac{L^4}{9} + \sum_{n=1}^{\infty}\frac{8L^4}{n^4\pi^4}$$

Equating the two sides of the theorem, we therefore get:

$$\sum_{m=1}^{\infty}\frac{1}{m^4} = \left(\frac{\pi^4}{8}\right)\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{90}$$

This is a series that converges faster than the ones we obtained directly from the series at special values of x.

Example 1.5.8: Let $f(x) = x\ , -\pi \leq x < \pi$. We found that the Fourier coefficients of f are:

$$a_n = 0\ \ ; and\ b_n = \frac{2(-1)^{n+1}}{n}.$$

By Eq.(*S*), real version of Parseval's thm.) we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \frac{4}{n^2}.$$

Evaluating the integral on the left and dividing both sides by 4, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad (compare\ to\ Example\ 1.4.3)$$

1.6. Differentiation and Integration of Fourier Series

Question: Can a Fourier series be differentiated term-by-term ? The answer is No, in general.

1.6.1 Fourier series and derivatives: Suppose that f is a $2L$ – periodic, piecewise smooth, and continuous function such that f' is also piecewise smooth. Let $a_n\ ,\ b_n$ denote the Fourier coefficients of f and $a_n'\ ,\ b_n'$ those of f', then

$$a_0' = 0\ , \qquad a_n' = b_n \frac{n\pi}{L}\ , \qquad and \qquad b_n' = -a_n \frac{n\pi}{L}$$

Notice, to compute the Fourier coefficients of f', evaluate the integrals by parts and use $f(L) = f(-L)$.

1.6.2 Term-by-term differentiation of Fourier series:

Suppose that f is a $2L$ – periodic, piecewise smooth, and continuous function such that f' is also piecewise smooth.

Then the Fourier series of f' is obtained from the Fourier series of f by differentiating term-by-term. That is, under the stated conditions, if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

then

$$f'(x) = \sum_{n=1}^{\infty} \left(-n a_n \frac{\pi}{L} \sin \frac{n\pi x}{L} + n b_n \frac{\pi}{L} \cos \frac{n\pi x}{L} \right).$$

Note that, since f' is piecewise smooth, it has a Fourier series expansion.

Remark 1.6.2.1: In most cases in this lectures, f and f' are piecewise smooth. Thus according to **subsection 1.6.2** , to differentiate term-by-term the Fourier series in these cases, **it is enough to check that f is continuous**. It is important to note that if f fails to satisfy some of the assumptions of **1.6.2** , then we cannot in general differentiate the series term-by-term.
According to the ***Remark 1.6.2.1***, we give the following Theorem.

Theorem 1.6.2.2 (***Differentiation of Fourier Series***): Let $f(x)$ be a continuous function with a piecewise continuous derivative in the interval $[-\pi, \pi]$, and let $f(\pi) = f(-\pi)$. Then, if the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

termwise differentiation of this series is permissible and

$$f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx} (a_n \cos nx + b_n \sin nx)$$

for $-\pi < x < \pi$, except at points where $f'(x)$ is discontinuous.

Example 1.6.2.3: Deduce the Fourier series of $f(x) = x$ in $[-\pi, \pi]$ from the Fourier series of $f(x) = \pi^2 - x^2$. Give an example where termwise differentiation of a Fourier series is not permissible.

Solution: We have seen that the Fourier series of $f(x) = \pi^2 - x^2$ in $[-\pi, \pi]$ is

$$\pi^2 - x^2 = \frac{2}{3}\pi^2 + 4\left(\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \cdots \right)$$

As $f(\pi) = 0 = f(-\pi)$ and $f(x)$ and $f'(x)$ are continuous, ***Theorem 1.6.2.2*** is applicable so that differentiation of the above result yields

$$-2x = -4\left(sin\, x - \frac{1}{2} sin\, 2x + \frac{1}{3} sin\, 3x - \cdots\right)$$

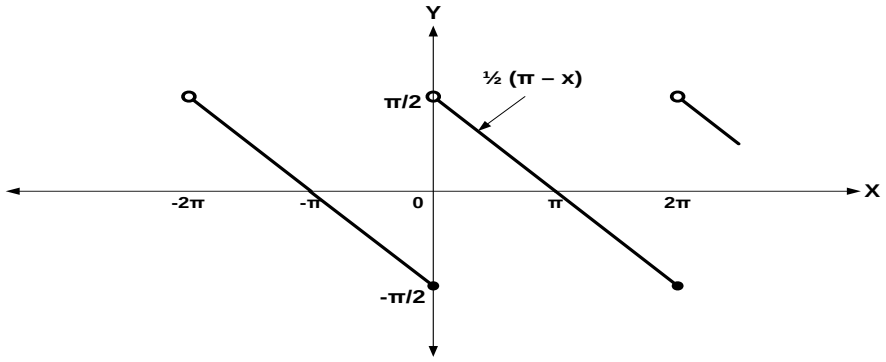
whence

$$x = 2\left(sin\, x - \frac{1}{2} sin\, 2x + \frac{1}{3} sin\, 3x - \cdots\right) \quad for \quad -\pi < x < \pi.$$

A direct calculation easily verifies that this is the correct Fourier expansion of the function $f(x) = x$ for $-\pi \leq x \leq \pi$.

On the other hand, consider the sawtooth function shown in the **following Figure**, is determined by the formulas

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x) & if\ 0 < x \leq 2\pi \\ f(x + 2\pi) & otherwise \end{cases}$$



Sawtooth function

The Fourier series of the sawtooth function is $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$. This is because,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\, dx = 0;$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\, cos\, nx\, dx = \frac{1}{2\pi}\left\{\int_0^{2\pi} \pi\, cos\, nx\ dx - \int_0^{2\pi} x\, cos\, nx\ dx\right\} = 0;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\, dx = \frac{1}{2\pi}\left\{\int_0^{2\pi} \pi \sin nx\ dx - \int_0^{2\pi} x \sin nx\ dx\right\}$$

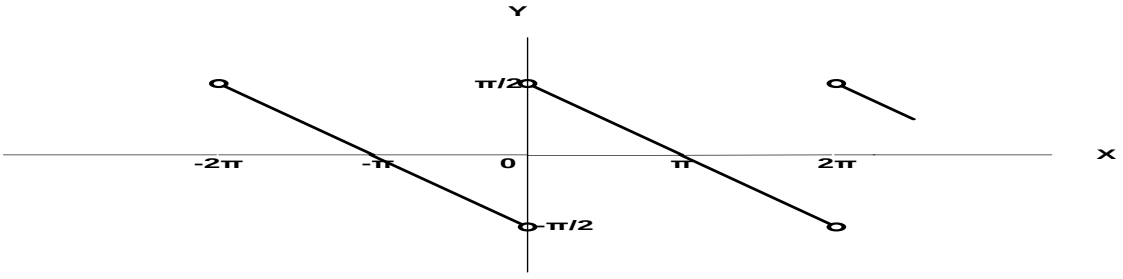
$$b_n = -\frac{1}{2\pi}\int_0^{2\pi} x \sin nx\ dx = \frac{1}{2\pi}\left\{\frac{-1}{n^2}\sin nx + \frac{x}{n}\cos nx\right\}\Big|_0^{2\pi} = \frac{1}{2\pi}\frac{2\pi}{n} = \frac{1}{n}.$$

Substituting these values of a_0 , a_n , $and\ b_n$ into Eq(*A*), we obtain $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ as the Fourier series of f.

Note that the Fourier series of f converges to $f(x)$ at each point *x* where f is continuous. In particular, for $0 < x < 2\pi$, we have

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{1}{2}(\pi - x).$$

At the points of discontinuities ($x = 2k\pi$, $k = 0, \pm1, \pm2, \ldots$), the series converges to 0. The graph of the Fourier series $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ is shown in the **<u>following Figure</u>**. It agrees with the graph of the function, except at the points of discontinuities.



The graph of the Fourier series of the sawtooth function *f* coincides with the graph of the function *f*, except at the points of discontinuity.

Now, <u>show that the function represented by this Fourier series satisfies all the hypotheses of ***1.6.2*** , except that it fails to be continuous</u>. In addition, we know that for all *x*, $\lim_{n\to\infty}\cos nx \neq 0$ and $\sum_{n=1}^{\infty}\cos nx$ is divergent. Thus, the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ <u>cannot</u> be differentiated term – by – term.

1.6.3 Fourier series and integration:

Suppose that f is a $2L$ – periodic, piecewise smooth function with the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and define an antiderivative of f by

$$F(x) = \int_0^x f(t)\, dt.$$

Then, F is a 2L – periodic function iff $\int_0^{2L} f(t)\, dt = 0$, and the Fourier series of F is

$$F(x) = A_0 + \sum_{n=1}^{\infty} \left(\frac{-L}{n\pi} b_n \cos \frac{n\pi x}{L} + \frac{L}{n\pi} a_n \sin \frac{n\pi x}{L} \right), where\ A_0 = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Hence, as long as F is periodic, with no further assumptions on f other than piecewise smoothness, we can get the Fourier series of F by integrating term-by-term the Fourier series of f.

Theorem 1.6.3.1 (Integration of Fourier Series): If $f(x)$ is an integrable piecewise continuous function in the interval $[-\pi, \pi]$ with the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

Then

$$\int_{x_1}^{x} f(t)\, dt = \frac{a_0(x - x_1)}{2} + \sum_{n=1}^{\infty} \int_{x_1}^{x} (a_n \cos nt + b_n \sin nt)\, dt$$

Example 1.6.3.2: Discuss the result of termwise integration of the Fourier series of the function

$$f(x) = \begin{cases} -1 & for\ -\pi < x < 0 \\ 1 & for\ \ 0 \le x \le \pi \end{cases}$$

over the interval $[-\pi, x]$ with $-\pi < x \le \pi$.

Solution: The Fourier series of $f(x)$ is

$$f(x) = \frac{4}{\pi}\left(sin\, x + \frac{1}{3} sin\, 3x + \frac{1}{5} sin\, 5x + \cdots\right) \ldots\ldots\ldots\ldots\ldots (a)$$

we now need to apply ***Theorem 1.6.3.1*** with $x_1 = -\pi$ and $x_2 = x$ and, in doing so, because of discontinuity in $f(x)$, to consider the two cases $-\pi < x < 0$ and $0 \leq x \leq \pi$.

(i) $-\pi < x < 0$. From ***Theorem 1.6.3.1*** we have

$$\int_{-\pi}^{x} (-1)\, dt = \frac{4}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{x} \frac{sin[(2n-1)t]}{(2n-1)}\, dt$$

or

$$-(x+\pi) = -\frac{4}{\pi}\left\{\sum_{n=1}^{\infty} \frac{cos[(2n-1)t]}{(2n-1)^2}\right\}\Bigg|_{-\pi}^{x}$$

$$= -\frac{4}{\pi}\left\{\left(cos\, x + \frac{cos\, 3x}{3^2} + \frac{cos\, 5x}{5^2} + \cdots\right) + \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)\right\}.$$

The numerical series has the sum $\pi^2/8$, so that this result simplifies to

$$-x = \frac{1}{2}\pi - \frac{4}{\pi}\left(cos\, x + \frac{cos\, 3x}{3^2} + \frac{cos\, 5x}{5^2} + \cdots\right) \quad for\ -\pi < x < 0 \ldots\ldots (b)$$

(ii) $0 \leq x \leq \pi$. Here, as a result of an application of ***Theorem 1.6.3.1***, we have

$$\int_{-\pi}^{0} (-1)\, dt + \int_{0}^{x} 1\, dt = \frac{4}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{x} \frac{sin[(2n-1)t]}{(2n-1)}\, dt$$

which after simplification as before gives

$$x = \frac{1}{2}\pi - \frac{4}{\pi}\left(cos\, x + \frac{cos\, 3x}{3^2} + \frac{cos\, 5x}{5^2} + \cdots\right) \quad for\ \ 0 \leq x \leq \pi \ldots\ldots (c)$$

Results (*b*) and (*c*), taken together, comprise the series representing the function

$$f(x) = \begin{cases} -x & for\ -\pi < x < 0 \\ x & for\ \ 0 \leq x \leq \pi \end{cases}$$

This is simply the function $f(x) = |x|$.

2. The Fourier Transform and Its Applications

2.1 The Fourier Integral Representation

To help us understand the topics of this section, let us recall from **section 1.1** the Fourier series representation (Formulation)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right) \dots\dots\dots\dots\dots\dots\dots (A)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx \quad \dots\dots\dots\dots\dots\dots\dots (B)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\ , \qquad n = 1,2,3,\dots \qquad \dots\dots\dots\dots\dots\dots (D)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\ , \quad n = 1,2,3,\dots \qquad \dots\dots\dots\dots\dots (F)$$

Now suppose that f is defined on the entire real line **but is not periodic**. Can we represent f by a Fourier series? It turns out that we no longer have a Fourier series representation, but a Fourier integral representation. The answer is given by the following important theorem.

Theorem 2.1.1: Suppose that f is piecewise smooth on every finite interval and that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$. Then f has the following Fourier integral representation

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \quad -\infty < x < \infty \qquad (2)$$

where, for all $\omega \geq 0$,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \quad , \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt \qquad (3)$$

The integral in (2) converges to $f(x)$ if f is ***continuous*** at x and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Note the similarity between the Fourier integral (2) and the Fourier series (*A*). The sum in (A) is replaced by an integral in (2), and the integrals from –*L* to *L* that define the

Fourier coefficients are replaced by integrals from $-\infty\ to\ \infty$ in (3). Also, in (3), the Fourier coefficients are computed over a continuous range $\omega \geq 0$, whereas the Fourier coefficients of a periodic function are computed over a discrete range of values $n = 0,1,2, \ldots$

As with F.S., for ***Theorem 2.1.1*** to hold we impose sufficient conditions on f, including the condition

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

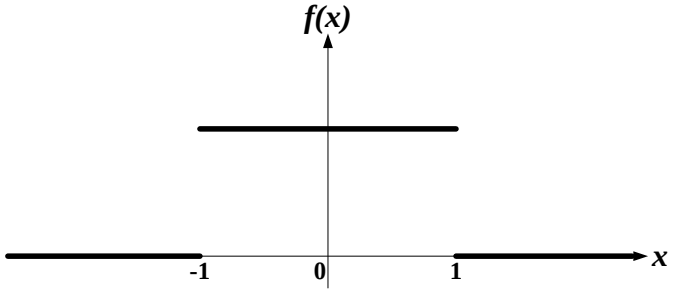
which is expressed by saying that f is **integrable on the entire real line**. This condition also ensures the existence of the improper integrals defining $A(\omega)$ and $B(\omega)$ in (3).

Example 2.1.2: (**A Fourier integral representation)**

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & if\ |x| \leq 1 \\ 0 & otherwise \end{cases}.$$

Solution: The graph of f is depicted bellow.



From (3),

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt = \frac{1}{\pi}\int_{-1}^{1} \cos\omega t\, dt = \left(\frac{\sin\omega t}{\pi\omega}\right)\Big|_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}.$$

Note that, we should treat the case $\omega = 0$ separately. However, as you can check, the formula that we obtained for $A(\omega)$ is valid in the limit as $\omega \to 0$.

Since $f(x)$ is even, then $B(\omega) = 0$. For $|x| \neq 1$ the function is ***continuous*** and

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega\cos\omega x}{\omega}\, d\omega. \quad (by\ Theorem\ 2.1.1)$$

For $x = \pm 1$, points of discontinuity of f, ***Theorem 2.1.1*** yields the value $\frac{1}{2}$ for the last integral. Note that, average of the left and right hand limits of f at $x = 1$ is $\frac{1+0}{2} = \frac{1}{2}$.
Thus we have the Fourier integral representation of f

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} 1 & if\ 0 \le x < 1 \\ 1/2 & if\ x = 1 \\ 0 & if\ x > 1 \end{cases} \qquad (1)$$

Therefore from (1)

$$\frac{1}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega$$

And

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} \pi/2 & if\ 0 \le x < 1 \\ \pi/4 & if\ x = 1 \\ 0 & if\ x > 1 \end{cases} \qquad (2)$$

Remark 2.1.3: In ***Example 2.1.2***, we have seen that $B(\omega) = 0$ as f is an even function. Similarly that, if f is an odd function then $A(\omega) = 0\ \forall\omega$. These observations simplify the computation of the Fourier integral representation of even and odd functions.
In addition, ***Theorem 2.1.1***, can be used to evaluate interesting **improper integral**. For example, setting $x = 0$ in the integral representation of ***Example 2.1.2*** yields the important integral

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}, \qquad (4)$$

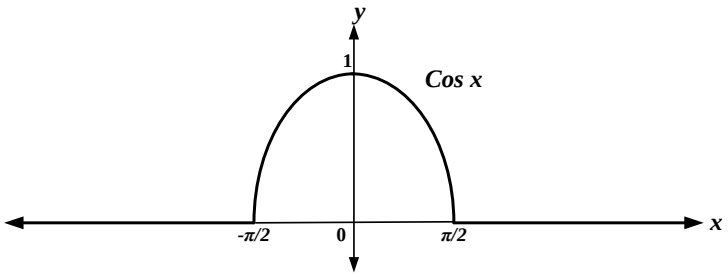
known as **Dirichlet's integral**.

Example 2.1.3: (**Computing Integrals via the Fourier Integral)**

Show that

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\cos\omega x\,d\omega = \begin{cases} \cos x & if\ |x| < \frac{\pi}{2} \\ 0 & if\ |x| > \frac{\pi}{2} \end{cases}.$$

Solution: Let $f(x)$ denote the function defined on the right side of this equality, as shown in the following Figure.



It is even and vanishes outside the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Thus, $B(\omega)=0\ \forall\omega\geq 0$, and

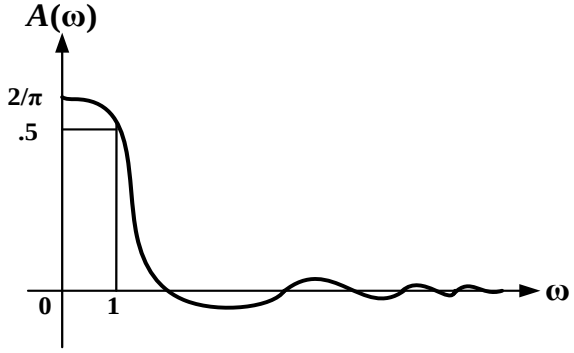
$$A(\omega)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos x\cos\omega x\,dx=\int_0^{\frac{\pi}{2}}[\cos(1+\omega)x+\cos(1-\omega)x]\,dx\ \ (Trig\ identity)$$

$$=\frac{1}{\pi}\left[\frac{sin[(1+\omega)\pi/2]}{1+\omega}+\frac{sin[(1-\omega)\pi/2]}{1-\omega}\right]\ \ for\ \omega\neq 1$$

$$=\frac{1}{\pi}\left[\frac{cos(\omega\pi/2)}{1+\omega}+\frac{cos(\omega\pi/2)}{1-\omega}\right]=\frac{2\,cos(\omega\pi/2)}{\pi(1-\omega^2)}.$$

The case $\omega=1$ should be treated separately. It yields $A(1)=\frac{1}{2}$ (check it !).

As the **following Figure** shows, the graph of $A(\omega)$ is **continuous** at $\omega=1$. In fact, you can check that $\lim_{\omega\to 1}A(\omega)=A(1)=\frac{1}{2}$.



Now, using (2), i.e.

$$f(x)=\int_0^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]\,d\omega\quad -\infty<x<\infty \qquad (2)$$

we get,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{cos(\omega\pi/2)}{1-\omega^2}\cos\omega x \; d\omega.$$

Replacing $f(x)$ by its formula, we obtain the desired identity. That is,

$$\begin{cases} \cos x & if \;\; |x| < \dfrac{\pi}{2} \\ 0 & if \;\; |x| > \dfrac{\pi}{2} \end{cases} = \frac{2}{\pi}\int_0^{\infty} \frac{cos(\omega\pi/2)}{1-\omega^2}\cos\omega x \; d\omega.$$

■ **Partial Fourier Integrals and the Gibbs Phenomenon**

In analogy with the partial sums of Fourier series, we define the partial Fourier integral of f by

$$S_v(x) = \int_0^{v} [A(\omega)\, cos\, \omega x + B(\omega)\, sin\, \omega x]\, d\omega \quad (for\ v > 0) \qquad (5)$$

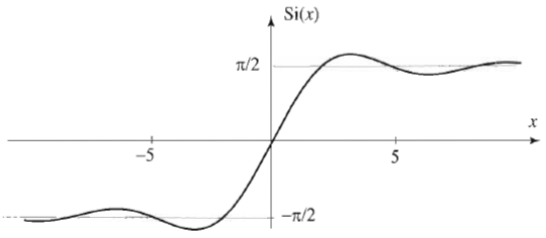
where $A(\omega)$ and $B(\omega)$ are given by (3). With this notation, ***Theorem 2.1.1*** states

$$\lim_{v\to\infty} S_v(x) = \frac{f(x+) + f(x-)}{2}.$$

Like F.S., near a point of discontinuity, the Fourier integral exhibits a Gibbs phenomenon. To illustrate this phenomenon, we introduce the **sine integral function**

$$Si(x) = \int_0^{x} \frac{sin\, t}{t}\, dt, \qquad -\infty < x < \infty \qquad (6)$$

The following Figure is the graph of $Si(x)$.



From (4), it follows that

$$\lim_{x\to\infty} Si(x) = \int_0^{\infty} \frac{sin\, t}{t}\, dt = \frac{\pi}{2} \qquad (7).$$

Example 2.1.4: (**Gibbs Phenomenon for Partial Fourier Integrals)**

Show that the partial Fourier integral of the function

$$f(x) = \begin{cases} 1 & if\ |x| \le 1 \\ 0 & otherwise \end{cases}.$$

can be written as

$$S_v(x) = \frac{1}{\pi}\left[Si\big(v(1+x)\big) + Si\big(v(1-x)\big)\right] \qquad (a)\,.$$

Solution: We have from ***Example 2.1.2*** and (5)

$$S_v(x) = \frac{2}{\pi}\int_0^v \frac{sin\,\omega\, cos\,\omega x}{\omega}\,d\omega.$$

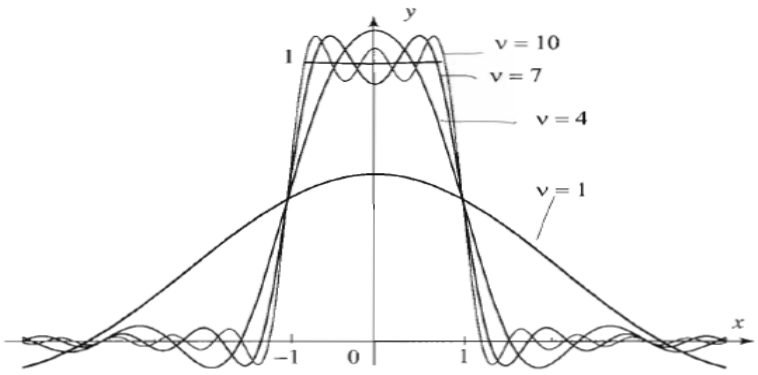
$$= \frac{1}{\pi}\int_0^v \frac{sin\,\omega(1+x)}{\omega}\,d\omega + \frac{1}{\pi}\int_0^v \frac{sin\,\omega(1-x)}{\omega}\,d\omega$$

Let $u = \omega(1+x)$ in the first integral and $u = \omega(1-x)$ in the second.

$$= \frac{1}{\pi}\left[\int_0^{v(1+x)} \frac{sin\,u}{u}\,du + \int_0^{v(1-x)} \frac{sin\,u}{u}\,du\right]$$

$$= \frac{1}{\pi}\left[Si\big(v(1+x)\big) + Si\big(v(1-x)\big)\right] \quad (by\ 6)$$

In the **following Figure**, we have plotted the graph of $S_v(x)$ for $v = 1, 4, 7, 10$, using the sine integral function and the formula given by (a). **Note the Gibbs phenomenon** at the points of discontinuity of f where the partial integrals overshoot their limiting values.



2.2 The Fourier Transform (F.T.)

In this section, we develop the ***Fourier transform*** **(F.T.)** and ***it's inverse*** as well as discuss some of its basic properties. The Fourier transform can be thought of as a continuous form of Fourier series. A Fourier series decomposes a signal on [-π,π] into components that vibrate an integer frequencies. By contrast, the Fourier transform decomposes a signal defined on an **infinite** time interval into a λ-frequency component, where λ can be any real (or even complex) number.

We will use the complex exponential function to write the Fourier integral representation of **section 2.1** in complex form. This new representation features an important pair of transforms: ***the Fourier transform and its inverse Fourier transform***.

Consider a continuous piecewise smooth integrable function f. Starting with the Fourier integral representation (***Theorem 2.1.1***), we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\,dt\,d\omega$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega\,(x-t)\,dt\,d\omega$$

$$\cos u = \frac{1}{2}\left(e^{iu} + e^{-iu}\right)$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\left(e^{i\omega(x-t)} + e^{-i\omega(x-t)}\right)dt\,d\omega$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\omega(x-t)}\,dt\,d\omega + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega(x-t)}\,dt\,d\omega\,.$$

If we change ω to $-\omega$ in the second term and adjust the limits on ω from $-\infty\ to\ 0$, we obtain, after adding the two integrals,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\omega(x-t)}\,dt\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \overbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt}^{\hat{f}(\omega)}\, d\omega \,.$$

This is the **complex form of the Fourier integral representation**, which features the following transform pairs:

Fourier Transform **(F.T.)**	$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \quad -\infty < \omega < \infty \quad (1)$

and,

Inverse Fourier Transform (since it describe $f(x)$ as an Integral involving the F.T. of f)	$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega \quad -\infty < x < \infty \quad (2)$

Remark 2.2.1: There are other conventions for the Fourier transform. For example, we could choose $\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$, and then the inverse Fourier transform becomes $f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$. In the definition of $\hat{f}$ we have used x as a variable of integration, instead of t. The symbols $\mathcal{F}(f)(\omega)$ and $\mathcal{F}^{-1}(f)(x)$ are also used to denote the Fourier transform and its inverse, respectively.

Sometimes, to be more specific, we will write $\mathcal{F}\big(f(x)\big)(\omega)$ instead of $\mathcal{F}(f)(\omega)$.

According to ***Theorem 2.1.1*** of section 2.1, if f is **not continuous** at x, the left side of (2) to be replaced by

$$\frac{f(x+) + f(x-)}{2} \,.$$

The integral for the inverse Fourier transform may not exist as a two-sided improper integral; in general, this integral should be computed as a Cauchy principle value:

$$f(x) = \frac{1}{\sqrt{2\pi}} \lim_{a \to \infty} \int_{-a}^{a} e^{i\omega x}\, \hat{f}(\omega)\, d\omega \,.$$

Putting $\omega = 0$ in (1), we find that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx\,.$$

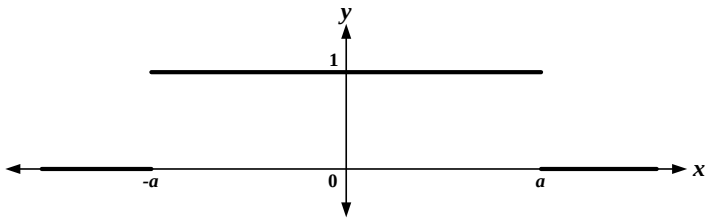
Thus, the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the *x*-axis, multiplied by the factor of $\frac{1}{\sqrt{2\pi}}$.

Example 2.2.1: (a) Find the Fourier transform of the function $f(x)$ given by

$$f(x) = \begin{cases} 1 & if\ |x| < a \\ 0 & if\ |x| > a \end{cases}.$$

What is $\hat{f}(0)$? (b) Express f as an inverse Fourier transform.

Solution: (a) The graph of f is depicted bellow.



For $\omega \neq 0$ we have

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1).\, e^{-i\omega x}\, dx$$

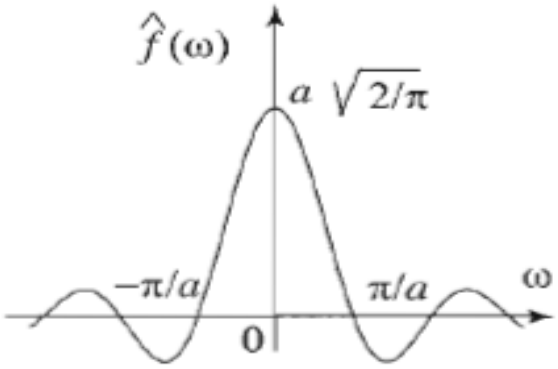
$$= \frac{-1}{\sqrt{2\pi}} \frac{1}{i\omega}\, e^{-i\omega x}\Big|_{-a}^{a} = \sqrt{\frac{2}{\pi}}\, \frac{\sin a\omega}{\omega}.$$

For $\omega = 0$ we have

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dx = a\sqrt{\frac{2}{\pi}}.$$

Since $\lim_{\omega \to 0} \hat{f}(\omega) = \lim_{\omega \to 0} \sqrt{\frac{2}{\pi}}\, \frac{\sin a\omega}{\omega} = a\sqrt{\frac{2}{\pi}} = \hat{f}(0)$, it follows that $\hat{f}(\omega)$ is continuous at 0 (**see the following Figure**), and we may write

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\, \frac{\sin a\omega}{\omega} \quad \forall\ \omega.$$



(b) To be express f as an inverse Fourier transform, we use (2) and get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin a\omega}{\omega} e^{i\omega x}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega} e^{i\omega x}\, d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sin\, a\omega}{\omega} (cos\, \omega x + isin\, \omega x)\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{cos\, \omega x\, sin\, a\omega}{\omega} d\omega$$

because $sin\, \omega x\, sin\, a\omega$ is an odd function of ω and so its integral is zero.

Note that, when $a = 1$, this representation coincides with the integral representation we found in ***Example 2.1.2*** of section 2.1.

In addition, the resulting Fourier transform is continuous on the entire real line even though the function has jump discontinuities at $x = \pm a$. In fact, it can be shown that the Fourier transform of an integrable function is **always continuous**.

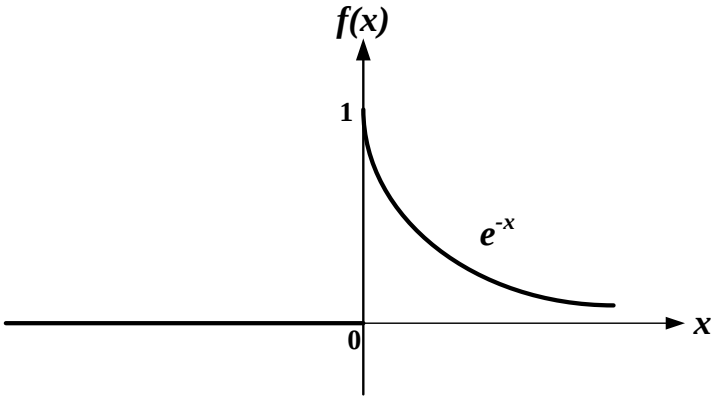
Remark 2.2.2: Let us recall that if $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$. In particular, if $z = e^{-ix\omega}$, then

$$\left|e^{-ix\omega}\right| = |cos\, \omega x - isin\, \omega x| = \sqrt{cos^2(x\omega) + sin^2(x\omega)} = 1.$$

Example 2.2.3: Find the Fourier transform of the function $f(x)$ given by

$$f(x) = \begin{cases} e^{-x} & if\ x > 0 \\ 0 & if\ x \leq 0 \end{cases}.$$

Solution: The graph of f is depicted bellow.



We have

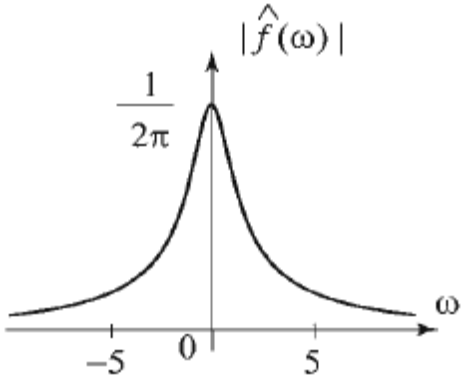
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (e^{-x}).\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(1+i\omega)}\, dx$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{1}{(1+i\omega)} \left((e^{-x}).\, e^{-i\omega x} \right) \Big|_{0}^{\infty}$$

Since $\left|e^{-ix\omega}\right| = 1$, it follows that $\lim_{x\to\infty} \left|e^{-x(1+i\omega)}\right| = \lim_{x\to\infty} e^{-x} = 0$, and so

$$\hat{f}(\omega) = \frac{-1}{\sqrt{2\pi}} \frac{1}{(1+i\omega)} = \frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)}.$$

The **following Figure** shows the graph of absolute value of $\hat{f}$. Here again, it is worth noting that $\hat{f}$ and $\left|\hat{f}\right|$ are both **continuous** even though f is not.



Note that, this example illustrates a noteworthy fact that the Fourier transform may be ***complex-valued*** even though ***the function is real-valued***. Also, the Fourier transform is ***continuous*** but ***not integrable***, not even as an improper two-sided integral. In this case, the integral in (2) for the inverse Fourier transform should be computed as a Cauchy principle value. Indeed, we can check that, at $x = 0$, we do have

$$\frac{1}{2} = \frac{f(0+) + f(0-)}{2}$$

and the Cauchy principle value of the inverse Fourier transform yields

$$\lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \hat{f}(\omega)\, d\omega = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1 - i\omega}{\sqrt{2\pi}(1+\omega^2)}\, d\omega$$

$$= \frac{1}{2\pi} \lim_{a\to\infty} \left[\int_{-a}^{a} \frac{1}{1+\omega^2}\, d\omega - i \overbrace{\int_{-a}^{a} \frac{\omega}{1+\omega^2}\, d\omega}^{=0} \right] = \frac{1}{2}.$$

Remark 2.2.4: As defined in (1), the inverse Fourier transform takes a function f and produces a new function $\hat{f}$, and the inverse transform recovers the original function f from $\hat{f}$. This process makes of transform pairs a powerful tool in solving partial differential equations.

As we will see in the next sections, the idea is to "Fourier transform" a given equation into one that may be easier to solve. After solving the transformed equation involving $\hat{f}$, we recover the solution of the original problem with the inverse transform. To assist us in handling transformed equations, we develop the operational properties of the Fourier transform.

Example 2.2.5: Compute the Fourier transform of the following function:

$$f(t) = \begin{cases} cos\, 3t & -\pi \le t \le \pi \\ 0 & otherwise \end{cases}.$$

Solution: Since f is an even function, only the cosine part of the transform contributes.

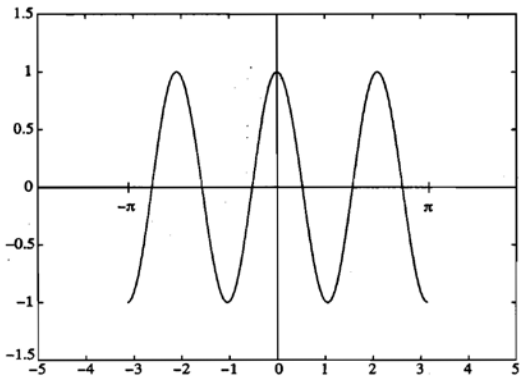
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos 3t \cos \lambda t\, dt = \frac{\sqrt{2}\,\lambda \sin \lambda\pi}{\sqrt{\pi}(9 - \lambda^2)}.$$

The above integral is obtained by using the following identities:

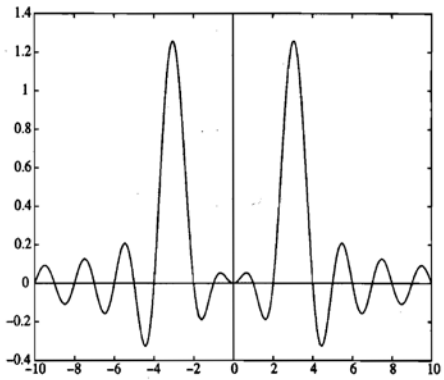
$$cos(u+v) = cos\, u\, cos\, v - sin\, u\, sin\, v \;;\; and \;\; cos(u-v) = cos\, u\, cos\, v + sin\, u\, sin\, v.$$

$$cos\, u\, cos\, v = \frac{1}{2}[cos(u+v) + cos(u-v)].$$

with $u = 3t$ and $v = \lambda t$, and then integrate. The following Figures show $f(t)$ and a graph of $\hat{f}$.



Plot of cos 3t



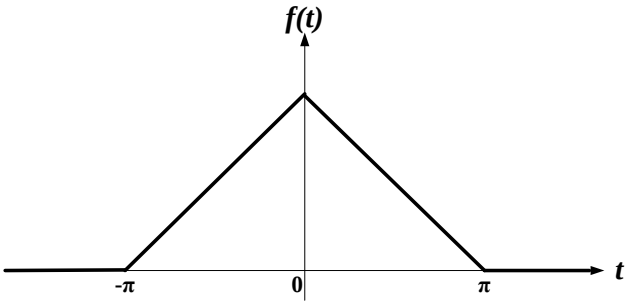
Fourier Transform of cos 3t

Note that the Fourier transform peaks at $\lambda = 3\ and\ -3$. This behavior should be expected since $f(t) = \cos 3t$ vibrates with frequency 3 on the interval $[-\pi, \pi]$.

Example 2.2.6: Find the Fourier transform of the triangular wave $f(t)$ where

$$f(t) = \begin{cases} t + \pi & -\pi \le t \le 0 \\ \pi - t & 0 < t \le \pi \\ 0 & otherwise \end{cases}.$$

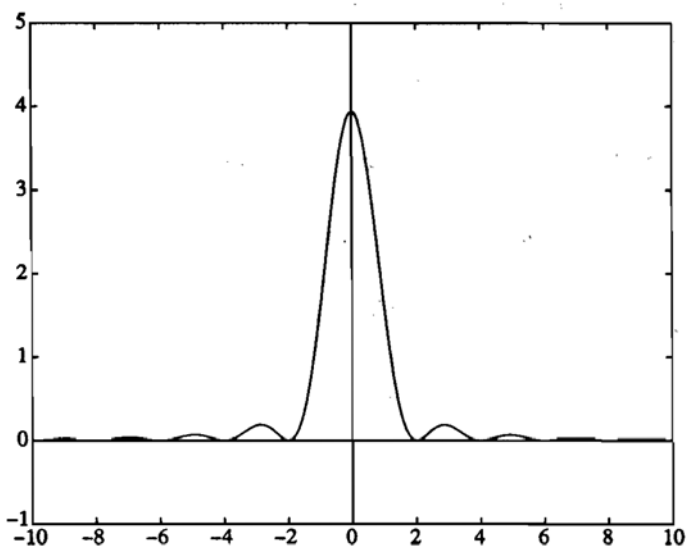
Solution: This function is even (see the following Figure),



and so it's Fourier transform is given by (note that, this function is even),

$$\hat{f}(\lambda) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t) \cos \lambda t \, dt = \frac{2}{\sqrt{2\pi}} \underbrace{\int_0^{\pi} (\pi - t) \cos \lambda t \, dt}_{using\ integration\ by\ parts} = \sqrt{\frac{2}{\pi}} \frac{(1 - cos\, \lambda\pi)}{\lambda^2}.$$

The graph of $\hat{f}$ is given in the following Figure.



Fourier transform of the triangular wave

Example 2.2.7: Find the Fourier transform of the boxcar function given by

$$f_1^1(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & othewise \end{cases}.$$

Express your answer as real function. Generalized your calculations to compute the Fourier transform $\hat{f}_a^A(\lambda)$ for the general boxcar function given by:

$$f_a^A(t) = \begin{cases} A & -a < t < a \\ 0 & othewise \end{cases}$$

The boxcar function is said to be *normalized* if $A = \frac{1}{2a}$ so that the total area under the function is 1.

We shall denote the normalized boxcar function that is non-zero on interval [-a, a] by f_a. **Fig.1** shows the boxcar function. Sometimes the values of f_a at $x = a$ and $x = -a$ are defined to be $\frac{1}{4a}$. For example, if a = $\frac{1}{2}$; the following graph represents f_a (**Fig.2**). Consider how changes in value of ***a*** impact the shape and values of $\hat{f}_a(\lambda)$.

Solution: $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt$. Since $f(t) = 0$ for $t < -1$ and $t > 1$, and $f(t) = 1$ for $-1 \le t \le 1$, we have that $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1.e^{-i\lambda t} dt = \frac{-1}{\sqrt{2\pi} i\lambda}\, e^{-i\lambda t}\Big|_{-1}^{1} = \frac{-1}{\sqrt{2\pi} i\lambda}\left(e^{-i\lambda} - e^{i\lambda}\right)$. Use the Euler's formula, we obtain

$$\hat{f}(\lambda) = \frac{-1}{\sqrt{2\pi} i\lambda}\left[cos(-\lambda) + i\, sin(-\lambda) - cos(\lambda) - i\, sin\,(\lambda)\right].$$

Using that $\cos(-\lambda) = \cos(\lambda)$, and $\sin(-\lambda) = -\sin(\lambda)$ since the cosine is even and the sine is odd function, the cosine terms cancel and we obtain that,

$$\hat{f}(\lambda) = \frac{-1}{\sqrt{2\pi}i\lambda}(-2i\ sin\ (\lambda)) = \frac{2}{\sqrt{2\pi}\lambda} sin\ (\lambda).$$

Similarly, if a boxcar functions of height A is nonzero on interval [-a,a]; the Fourier transform is:

$$\hat{f}_a^A(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} Ae^{-i\lambda t} dt = \frac{A}{\sqrt{2\pi}i\lambda}\left(e^{-ia\lambda} - e^{ia\lambda}\right) = \frac{2A}{\sqrt{2\pi}\lambda}\left(\frac{e^{ia\lambda} - e^{-ia\lambda}}{2i}\right)$$

$$= \frac{2A}{\sqrt{2\pi}\lambda} sin\ (a\lambda).$$

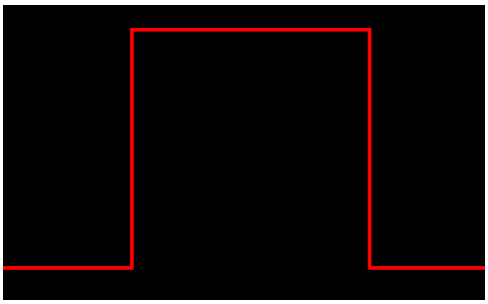


Fig.1: the boxcar $f(t)$, -a < t < a

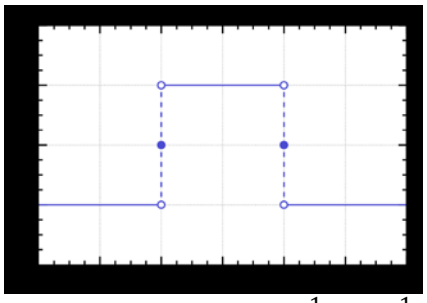
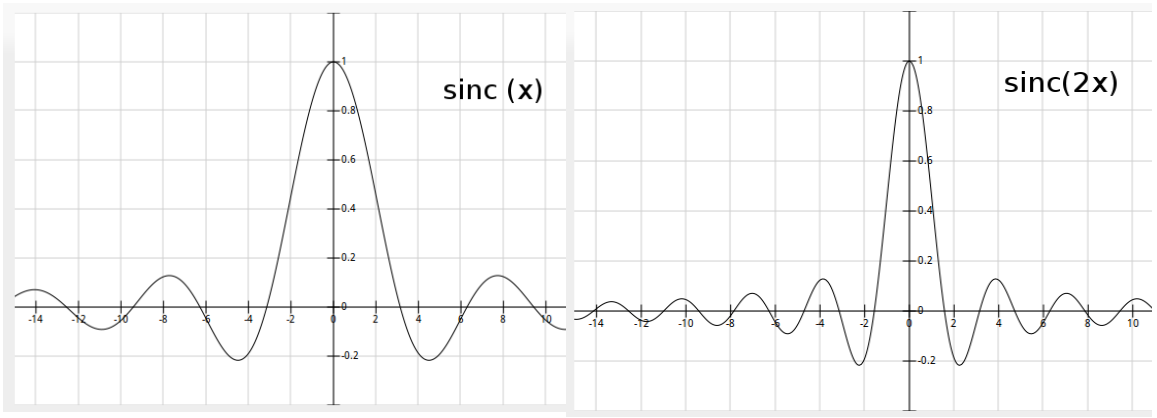
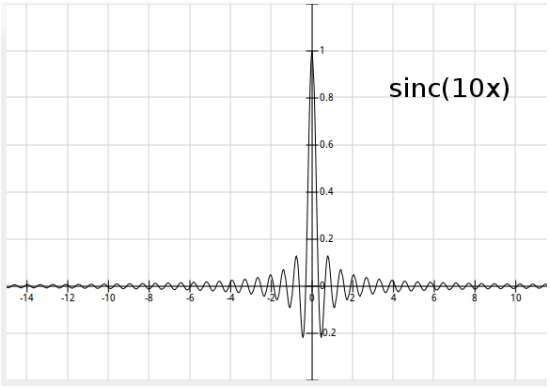


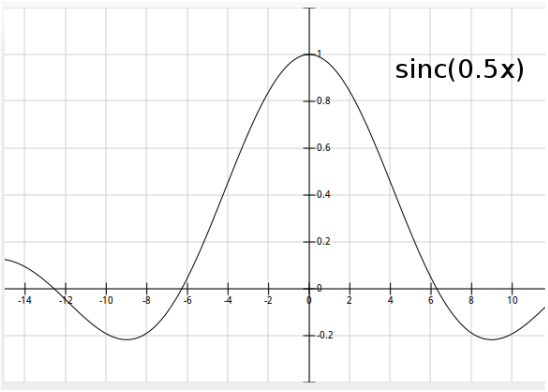
Fig.2: the boxcar $f(t)$, $-\frac{1}{2} < t < \frac{1}{2}$

Note that the function $\frac{\sin x}{x}$ is known as ***sinc*** function. That is; $sinc(x) = \frac{\sin x}{x}$.

Sometimes the normalized *sinc* function $\frac{\sin \pi x}{\pi x}$ is used. Note that $\lim_{x\to 0} sinc(x) = 1$, *and* $\lim_{x\to\infty} sinc(x) = 0$.







Using the ***sinc*** notation, we can represent the Fourier transform of the boxcar function $f_a^A(t)$ as the ***sinc*** function $\hat{f}_a^A(\lambda) = \frac{2aA}{\sqrt{2\pi}} sinc(a\lambda)$.

If the box function is normalized, $A = \frac{1}{2a}$, so $\hat{f}_a(\lambda) = \frac{1}{\sqrt{2\pi}} sinc(a\lambda)$. Since ***sinc***(0) = 1, the peak of this function has value $\frac{1}{\sqrt{2\pi}}$.

Let us compare the graphs of f_a and $\hat{f}_a$ for several different values of a. You can notice that larger values of a correspond to f_a having **smaller hight** and being **more spread out**. In this case, $\hat{f}_a$ has a narrow, sharp peak at λ = 0 and converges to 0 faster. If a is small, f_a has **large height** and very **narrow base**. In this case, $\hat{f}_a$ has very spread out peak around λ = 0 and the convergence to 0 is much slower.

Example 2.2.8: Find the inverse Fourier transforms of boxcar function.

Solution: The inverse transform of the general boxcar function can be computed by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} Ae^{i\lambda t}\, d\lambda = \frac{A}{\sqrt{2\pi} it} \left(e^{iat} - e^{-iat}\right).$$

To express this answer as a real function, use the Euler's formula and symmetries of sine and cosine functions similarly as in ***Example 2.2.7*** to obtain the following.

$$f(t) = \frac{A}{\sqrt{2\pi} it} [cos(at) + i\, sin(at) - cos(-at) - i\, sin\,(-at)]$$

$$f(t) = \frac{A}{\sqrt{2\pi} it} [cos(at) + i\, sin(at) - cos(at) + i\, sin(at)]$$

$$= \frac{A}{\sqrt{2\pi} it} (2i\, sin(at)) = \frac{2A}{\sqrt{2\pi} t} \left(sin(at) \frac{a}{a} \right)$$

$$= \frac{2A}{\sqrt{2\pi}t}\left(sin(at)\frac{a}{a}\right) = \frac{2aA}{\sqrt{2\pi}}\frac{sin\,(at)}{at} = \frac{2aA}{\sqrt{2\pi}}\,sinc(at).$$

Thus, the inverse transform of the boxcar function is the sinc function. This also illustrates that **the Fourier transform of the *sinc* function is the boxcar function**.

Example 2.2.9: Consider the Gaussian probability function (the "bell curve");

$$f(t) = Ne^{-at^2},$$

where *N* and *a* are constants. *N* determines the height of the peak and *a* determines how fast it decreases after reaching the peak. The Fourier transform of *f*(*t*) can be found to be

$$\hat{f}(\lambda) = \frac{N}{\sqrt{2a}}e^{\frac{-\lambda^2}{4a}}.$$

This is another Gaussian probability function. Examine how changes in ***a*** impact the graph of the transform.

Solution: If ***a*** is small, *f* is flattened. In this case the presence of ***a*** in the denominator of the exponent of $\hat{f}(\lambda)$will cause the $\hat{f}$ to be sharply peaked and the presence of ***a*** in the denominator of $\frac{N}{\sqrt{2a}}$ will cause $\hat{f}$ to have high peak value. If ***a*** is large, *f* is sharply peaked and $\hat{f}$ is flattened and with small peak value.

Note: In the literature, the alternative notation $\boldsymbol{\mathcal{F}[f](\lambda) = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx}$ is used for the Fourier transform of *f*, provided that the integral exists. This notation has advantages when discussing some of the operator theoretic properties of the Fourier transform. The Fourier operator $\boldsymbol{\mathcal{F}}$ should be thought of as a mapping whose domain and range are the space of complex-valued functions defined on the real line. The input of $\boldsymbol{\mathcal{F}}$ is a function, say *f*, and returns another function $\boldsymbol{\mathcal{F}[f] = \hat{f}(\lambda), as\ its\ output}$.

In a similar fashion, we define the inverse Fourier transform operator as:

$$\mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)\,e^{i\lambda x}d\lambda.$$

It is easy to show that $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$:

$$\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x);\ \ because$$

$$\mathcal{F}^{-1}[\mathcal{F}[f]](x) = \mathcal{F}^{-1}\left[\hat{f}\right](x)\ \ by\ definition\ of\ \mathcal{F} = \hat{f}\ .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}\,(\lambda) e^{i\lambda x} d\lambda = f(x)\,;\ \ by\ definition\ of\ \mathcal{F}^{-1}\left[\hat{f}\right]$$

Remark 2.2.10: the most important Fourier transform formula in practice is

$$\mathcal{F}\left[e^{-ax^2}\right](\lambda) = \hat{f}(\lambda) = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}}\,;$$

In particular,

$$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\lambda) = \hat{f}(\lambda) = e^{-\frac{\lambda^2}{2}}\ .$$

Proof:

$$\mathcal{F}\left[e^{-ax^2}\right](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2}\,.\,e^{-i\lambda x}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2 - \frac{\lambda^2}{4a}}\,dx$$

$$= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\lambda}{2a}\right)^2}\,dx = e^{-\frac{\lambda^2}{4a}} \int_{-\infty+\frac{i\lambda}{2a}}^{\infty+\frac{i\lambda}{2a}} \frac{1}{\sqrt{2\pi}} e^{-ay^2}\,dy$$

$$= \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}} \int_{-\infty+\frac{i\lambda}{2\sqrt{a}}}^{\infty+\frac{i\lambda}{2\sqrt{a}}} \frac{1}{\sqrt{\pi}} e^{-y^2}\,dy = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}}\,.$$

Here we have used the fact that

$$\int_{-\infty+\frac{i\lambda}{2\sqrt{a}}}^{\infty+\frac{i\lambda}{2\sqrt{a}}} e^{-y^2}\,dy = \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}\,.$$

The proof for the first equality can be found in any Complex Analysis textbook. The proof for the second equality is as follows:

We introduce a new variable *x*. Then

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \left[\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\right]^{\frac{1}{2}}$$

$$\left[\iint e^{-(x^2+y^2)}\,dxdy\right]^{\frac{1}{2}} = \left[\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta\right]^{\frac{1}{2}} = \left[\pi\int_0^{\infty} e^{-z}dz\right]^{\frac{1}{2}} = \sqrt{\pi}\,.$$

Symmetry considerations

The following tables display the ***formulas*** of Fourier and inverse Fourier transform for ***even*** and ***odd*** functions.

Fourier sine and cosine functions: Suppose that *f*(*t*) is defined just for t > 0. We can extend *f*(*t*) so that it is ***even***. Then we get the formula for $\hat{f}(\lambda)$ by using the formulas for ***even*** function above. $\hat{f}(\lambda)$is then called ***Fourier cosine transformation***.

Similarly, if we extend *f*(*t*) so that it is ***odd***, we get the formula for $\hat{f}(\lambda)$ same as for an ***even*** function above. $\hat{f}(\lambda)$ is then called ***Fourier sine transformation***.

$f(t)\ even$	$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\lambda t\,dt$
$\hat{f}(\lambda)\ even$	$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}(\lambda)\cos\lambda t\,dt$

$f(t)\ 0dd$	$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\lambda t\,dt$

$\hat{f}(\lambda)\ 0dd$	$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\lambda) \sin \lambda t \, dt$

Example 2.2.11: Find the Fourier transform and it's inverse of the box function (or rectangular wave).

$$f(t) = \begin{cases} 1 & -\pi < t < \pi \\ \frac{1}{2} & t = \pm\pi \\ 0 & otherwise \end{cases}.$$

Solution: Since *f*(*t*) is an even function, then

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos \lambda t \ dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos \lambda t \ dt = \frac{1}{\sqrt{2\pi}} \left(\frac{2 \sin \lambda\pi}{\lambda} \right) = \frac{2\pi}{\sqrt{2\pi}} \left(\frac{2 \sin \lambda\pi}{\lambda\pi} \right)$$
$$= \frac{2\pi}{\sqrt{2\pi}} (sinc\ \lambda) = \sqrt{2\pi}(sinc\ \lambda)$$

Thus $sinc\ \lambda$ is the Fourier transform of the box function. The inverse Fourier transform is

$$\int_{-\infty}^{\infty} sinc\ \lambda\, e^{i\lambda t}\, d\lambda = f(t);$$

$this\ is\ clear\ from\ complex\ variable\ theory\ or\ limit\ argument\ in\ calc.$

Note that $\|f\|^2 = \int_{-\infty}^{\infty} (f(t))^2 \ dt = 2\pi$; and $\int_{-\infty}^{\infty} (|sinc\ \lambda|)^2 d\lambda = 1.$

Note that the inverse Fourier transform converges to the ***midpoint of the discontinuity***, just as for Fourier series.

Remark 2.2.12: Not every function *f*(*x*) has a Fourier transform. A ***sufficient condition*** is that it be ***square-integrable***; that is, so that the following integral converges:

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

If in addition of being ***square-integrable***, the function is ***continuous***, then one also has the inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}\,(\lambda) e^{i\lambda x} d\lambda.$$

More generally, one has the ***Fourier inversion theorem***, which states that if *f*(*x*) is square-integrable, then the Fourier transform $\hat{f}(\lambda)$ exists and moreover

$$\int_{-\infty}^{\infty} \hat{f}\,(\lambda) e^{i\lambda x} d\lambda = \begin{cases} f(x) & if\ f\ is\ continuous\ at\ x \\ \frac{1}{2}[f(x+) + f(x-)] & otherwise \end{cases}.$$

In other words, at a point of discontinuity, the inverse transform produces the average of the left and right limiting values of the function *f*. This was also the case with the Fourier series. In any case, assuming that the function *f*(*x*) is such that its points of discontinuity are isolated, then the inverse transform will agree with *f*(*x*) everywhere but at the discontinuities.

Remark 2.2.13: consider the Fourier transform of a finite wave train:

$$f(x) = \begin{cases} \sin x & |x| \le 6\pi \\ 0 & otherwise \end{cases}.$$

This function is clearly square-integrable, with $\|f\|^2 = 6\pi\,, since$

$$\|f\|^2 = \int_{-6\pi}^{6\pi} (\sin x)^2\, dx = 6\pi.$$

The F.T. of f is given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-6\pi}^{6\pi} \sin x\ e^{-i\lambda x}\, dx$$

$$= \frac{1}{2i\sqrt{2\pi}} \int_{-6\pi}^{6\pi} \left(e^{ix} - e^{-ix}\right) e^{-i\lambda x}\, dx$$

$$= \frac{1}{2i\sqrt{2\pi}} \int_{-6\pi}^{6\pi} \left(e^{i(1-\lambda)x} - e^{-i(1+\lambda)x}\right) dx$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\frac{i}{\lambda - 1}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right) - \frac{i}{1+\lambda}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{i}{2i(\lambda - 1)}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right) - \frac{i}{2i(1+\lambda)}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{i.i}{2i^2(\lambda - 1)}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right) - \frac{i.i}{2i^2(1+\lambda)}\left(e^{-i\lambda 6\pi} - e^{i\lambda 6\pi}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2(1 - \lambda)}\left(e^{i\lambda 6\pi} - e^{-i\lambda 6\pi}\right) + \frac{1}{2(1+\lambda)}\left(e^{i\lambda 6\pi} - e^{-i\lambda 6\pi}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{1}{(1 - \lambda)} + \frac{1}{(1+\lambda)}\right)\left(\frac{e^{i\lambda 6\pi} - e^{-i\lambda 6\pi}}{2}\right)\right]$$

$$= \frac{i}{\sqrt{2\pi}} \left[\left(\frac{1}{(1 - \lambda)} + \frac{1}{(1+\lambda)}\right)\left(\frac{e^{i\lambda 6\pi} - e^{-i\lambda 6\pi}}{2i}\right)\right]$$

$$= \frac{i\ sin\ 6\pi\lambda}{\sqrt{2\pi}} \left[\frac{1}{1 - \lambda} - \frac{1}{1+\lambda}\right] = \sqrt{\frac{2}{\pi}}\ \left[\frac{i\ sin\ 6\pi\lambda}{(1 - \lambda^2)}\right].$$

We will not verify the inversion formula for this transform; although in this case the formula holds for all *x* since the original function is continuous.

■ Operational Properties

We shall investigate the behaviour of the Fourier Transform in connection with the common operations on functions: linear combination, translation, dilation, differentiation, multiplication by polynomials, and convolution.

Theorem 2.2.14: **(Some properties of FT and its inverse)**

Let f and g be differentiable functions defined on the real line with $f(t) = 0$ for large $|t|$. The following properties hold.

1. The Fourier transform and its inverse are **linear operators**. That is, for any constant ***c***,

$$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g] \ ; \ and \ \ \mathcal{F}[cf] = c\mathcal{F}[f].$$

$$\mathcal{F}^{-1}[f+g] = \mathcal{F}^{-1}[f] + \mathcal{F}^{-1}[g] \ ; \ and \ \ \mathcal{F}^{-1}[cf] = c\mathcal{F}^{-1}[f].$$

2. The Fourier transform of a product of f with t^n is given by

$$\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}.$$

3. The inverse Fourier transform of a product of f with λ^n is given by

$$\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) = -i^n \frac{d^n}{dt^n}\{\mathcal{F}^{-1}[f](t)\}.$$

4. The Fourier transform of an *n*th derivative is given by

$$\mathcal{F}[f^n](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda).$$

5. The inverse Fourier transform of an *n*th derivative is given by

$$\mathcal{F}^{-1}[f^n](t) = (-it)^n \mathcal{F}^{-1}[f](t).$$

6. The Fourier transform of a translation is given by

$$\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda).$$

7. The Fourier transform of a rescaling is given by

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f]\left(\frac{\lambda}{b}\right).$$

8. If $f(t) = 0$ for $t < 0$, then

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}Ł[f](i\lambda). \textit{where } Ł[f] \textit{ is the Laplace transform of } f \textit{ defined by}$$

$$Ł[f](s) = \int_0^\infty f(t)\, e^{-ts} dt.$$

Proof: **1.** The linearity of the Fourier transform follows from the linearity of integrals.

$$\mathcal{F}[f+g](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(t) + g(t)]\, e^{-i\lambda t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)\, e^{-i\lambda t} dt = \mathcal{F}[f](\lambda) + \mathcal{F}[g](\lambda).$$

The proof for $\mathcal{F}[cf] = c\mathcal{F}[f]$ is similar, as are the proofs for the corresponding facts for the inverse Fourier transform.

---

2. $\mathcal{F}[t^n f(t)](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n f(t)\, e^{-i\lambda t} dt$

Using the Eq.

$$t^n f(t) e^{-i\lambda t} = (i)^n \frac{d^n}{d\lambda^n}\{f(t) e^{-i\lambda t}\}; \quad we\ obtain,$$

$$\mathcal{F}[t^n f(t)](\lambda) = (i)^n \frac{d^n}{d\lambda^n}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt\right\} = (i)^n \frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}.$$

The corresponding property for the inverse Fourier transform is proved similarly.

4-5. For the FT of the n^{th} derivative of f, we have

$$\mathcal{F}[f^n(t)](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^n(t)\, e^{-i\lambda t} dt.$$

We now integrate by parts; that is

$$\int_{-\infty}^{\infty} udv = uv|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} vdu$$

With $u = e^{-i\lambda t}$ and $dv = f^n(t)dt$. As we will see, this process has the effect of transferring the derivatives from f^n to $e^{-i\lambda t}$. Since f vanishes at $+\infty$ and $-\infty$ by hypothesis, there are no boundary terms. So we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^n(t)\, e^{-i\lambda t} dt = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(n-1)}(t) \frac{d}{dt}\{e^{-i\lambda t}\}\, dt \qquad by\ parts$$

$$= (i\lambda) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(n-1)}(t) \frac{d}{dt}\{e^{-i\lambda t}\}\, dt.$$

Note that integration by parts has reduced the number of derivatives on f by one [f^nbecomes $f^{(n-1)}$]. We have also gained one factor of iλ. By repeating this process (n-1) additional times, we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^n(t)\, e^{-i\lambda t} dt = (i\lambda)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt = (i\lambda)^n \mathcal{F}[f](\lambda).$$

The proof for the corresponding fact for the inverse Fourier transform is similar.

6-7. The translation and rescaling properties can be combined into the following statement:

$$\mathcal{F}[f(bt - a)](\lambda) = \frac{1}{b} e^{\frac{-i\lambda a}{b}} \mathcal{F}[f]\left(\frac{\lambda}{b}\right).$$

This Eq. can be established by the following change of variables argument:

$$\mathcal{F}[f(bt-a)](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(bt-a)\, e^{-i\lambda t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-i\lambda\left(\frac{s+a}{b}\right)} \frac{ds}{b}$$

where the second equality follows from the change of variables $s = bt - a$, and so $t = \frac{s+a}{b}$; and $dt = \frac{ds}{b}$.

Rewriting the exponential on the right as $e^{-i\lambda\left(\frac{s+a}{b}\right)} = e^{\frac{-i\lambda a}{b}} e^{-i\frac{\lambda}{b}s}$; we obtain

$$\mathcal{F}[f(bt-a)](\lambda) = e^{\frac{-i\lambda a}{b}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-i\frac{\lambda}{b}s} \frac{ds}{b} = e^{\frac{-i\lambda a}{b}} \frac{1}{b} \mathcal{F}[f]\left(\frac{\lambda}{b}\right).$$

---

8. This part regarding the relationship of the Fourier and Laplace transforms follows from their definitions.

Remark 2.2.15: The property (**6)** is also called (***shifting property***). ***According to the shift theorem***, a shift of the variable t (a delay in time) has a simple effect on the Fourier transform. We would expect the magnitude of the Fourier transform $|\mathcal{F}[f](\lambda)|$ to stay the same, since shifting the original signal in time should not change the energy at any point in the spectrum. Hence the only change should be a phase shift in $\mathcal{F}[f](\lambda)$, and that's exactly what happens.

To compute the Fourier transform of $f(t\ +\ b)$ for any constant b, we have

$$\int_{-\infty}^{\infty} f(t+b)\, e^{-i\lambda t} dt = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-b)} du$$

$$(substituting\ u = t + b, the\ limits\ still\ go\ from -\infty\ to\ \infty.)$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} e^{i\lambda b} du = e^{i\lambda b} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du = e^{i\lambda b} \hat{f}(\lambda).$$

Notice that, as promised, the magnitude of the Fourier transform has not changed under a time shift because the factor out front has magnitude 1.

$$\left|e^{i\lambda b} \hat{f}(\lambda)\right| = \left|e^{i\lambda b}\right| \left|\hat{f}(\lambda)\right| = \left|\hat{f}(\lambda)\right|.$$

Remark 2.2.16: The property (**7)** is called (***scaling property***). ***According to the stretch* (*similarity*) *theorem***, how does the Fourier transform change if we ***stretch*** or ***shrink*** the variable in the time domain? More precisely, we want to know if we scale t to bt what happens to the Fourier transform of $f(bt)$. First suppose $b > 0$, then

$$\int_{-\infty}^{\infty} f(bt)\, e^{-i\lambda t} dt = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda\left(\frac{u}{b}\right)} \frac{1}{b} du$$

Substitution $u = bt$, the limits go the same way because $b > 0$.

$$= \frac{1}{b} \int_{-\infty}^{\infty} f(u)\, e^{-i\left(\frac{\lambda}{b}\right)u} e^{i\lambda b} du = \frac{1}{b} \mathcal{F}[f]\left(\frac{\lambda}{b}\right).$$

If $\boldsymbol{b < 0}$, the limits of integration are reversed when we make the substitution $u = bx$, and so the resulting transform is $(-\frac{1}{b})\mathcal{F}[f]\left(\frac{\lambda}{b}\right)$. Since $(-b)$ is positive when b is negative, we can combine the two cases and present the Stretch Theorem in its full glory:

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{|b|} \mathcal{F}[f]\left(\frac{\lambda}{b}\right).$$

This is also sometimes called the ***Similarity Theorem*** because changing the variable from x to ax is a change of scale, also known as a similarity.

Remark 2.2.17: There's an important observation that goes with the stretch theorem. Let's take b to be positive, just to be definite. If b is large (bigger than 1, at least) then the graph of $f(bt)$ is squeezed horizontally compared to $f(t)$. Something different is happening in the frequency domain, in fact in two ways. The Fourier transform is $\frac{1}{|b|}\mathcal{F}[f]\left(\frac{\lambda}{b}\right)$. If b is large then $\mathcal{F}[f]\left(\frac{\lambda}{b}\right)$ is ***stretched out*** compared to $\mathcal{F}[f](\lambda)$, rather than squeezed in.

Furthermore, multiplying by $\frac{1}{b}$, since the transform is $\frac{1}{|b|}\mathcal{F}[f]\left(\frac{\lambda}{b}\right)$, also squashes down the values of the transform.

The opposite happens if b is small (less than 1). In that case the graph of $f(bt)$ is stretched out horizontally compared to $f(t)$, while the Fourier transform is compressed ***horizontally*** and stretched ***vertically***. The phrase that's often used to describe this phenomenon is that a signal cannot be ***localized*** (meaning concentrated at a point) in both the time domain and the frequency domain.

Example 2.2.18: In this example, we illustrate the fourth property of the above theorem with the function

$$f(t) = \begin{cases} sin3t & -\pi \leq t \leq \pi \\ 0 & otherwise \end{cases}$$

Solution: We know that FT of f is given by:

$$\hat{f}(\lambda) = \frac{-3\sqrt{2}i\ sin\lambda\pi}{\sqrt{\pi}(9-\lambda^2)}$$

The derivative of f is $3cos3t\ on - \pi \leq t \leq \pi$, that is; $f'(t) = 3cos3t, -\pi \leq t \leq \pi$. therefore, the FT of $f'\ is\ given\ by$:

$$\widehat{f'}(\lambda) = \frac{3\sqrt{2}\lambda\ sin\lambda\pi}{\sqrt{\pi}(9-\lambda^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

The fourth property of the above theorem (with n = 1) states

$$\widehat{f'}(\lambda) = (i\lambda)\hat{f}(\lambda) = (-i\lambda)\frac{3\sqrt{2}i\ sin\lambda\pi}{\sqrt{\pi}(9-\lambda^2)}.$$

This expression for $\widehat{f'}(\lambda)$ agrees with Eq.(A).

Example 2.2.19: In this example, we illustrate the scaling property **(7)** of the above theorem

$$\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f]\left(\frac{\lambda}{b}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Solution: If $b > 1$, the graph of $f(bt)$ is a compressed version of the graph of f. The dominant frequencies of $f(bt)$ are larger than those of f by a factor of b. This behavior is illustrated nicely by the function in the above example

$$f(t) = \begin{cases} sin3t & -\pi \le t \le \pi \\ 0 & otherwise \end{cases}$$

whose graph is given in the ***Figure 1*** below. The graph of $f(2t)$ is given in the ***Figure 2*** below. Note that the frequency $f(2t)$ is double that of f.

Increasing the frequency of a signal has the effect of stretching the graph of its FT. In the preceding example, the dominant frequency of $f(t)$ is 3 whereas the dominant frequency of $f(2t)$ is 6. Thus, the maximum value of $|\hat{f}(\lambda)|$ occurs at λ = 3 (see **Fig.3**), whereas the maximum value of FT of $f(2t)$ occurs at λ = 6 (see **Fig.4**). Thus the latter graph is obtained by stretching the former graph by a factor of 2.

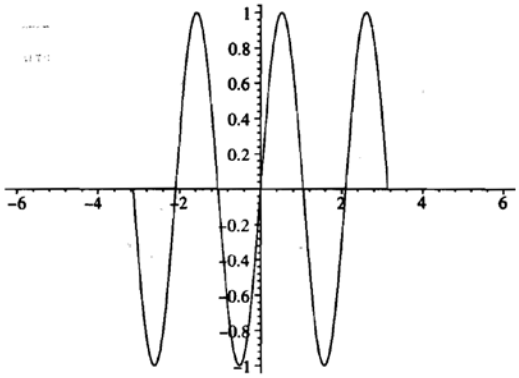


Fig.1: plot of sin(3*t*), $-\pi \le t \le \pi$

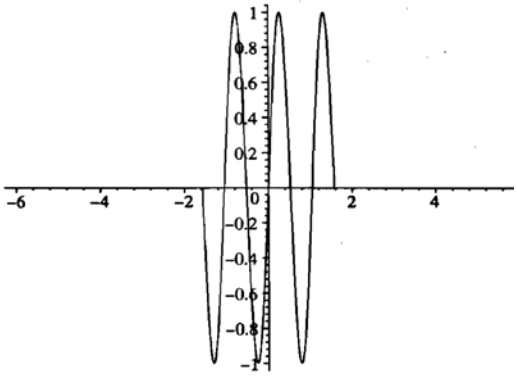


Fig.2: plot of sin(6*t*), $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

Note also that the graph of $\hat{f}(\frac{\lambda}{2})$ is obtained by stretching the graph of $\hat{f}(\lambda)$ by a factor of 2.

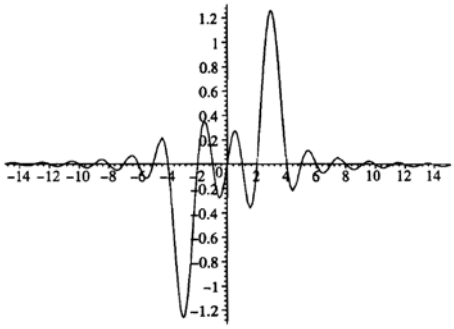


Fig.3: plot of FT of sin(3*t*)

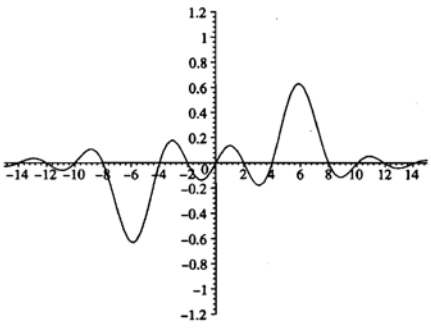


Fig.4: plot of FT of sin(6*t*)

This discussion illustrates the following geometrical interpretation of Eq.(*B*) in the case where $b > 1$:

"Compressing the graph of f speeds up the frequency and therefore stretches the graph of $\hat{f}(\lambda)$.

If $0 < b < 1$, then the graph of $f(bt)$ is stretched, which slows the frequency and therefore compresses the graph of $\hat{f}(\lambda)$.

■ Fourier Transform of a Convolution

How can we use one signal to modify another? Some of the properties of the Fourier transform that we have already derived can be thought of as addressing this question. The easiest is the result on additivity, according to which

$$\mathcal{F}[f + g] = \mathcal{F}[f] + \mathcal{F}[g]$$

Adding the signal $g(t)$ to the signal $f(t)$ adds the amounts $\mathcal{F}$g(λ) to the frequency components $\mathcal{F}f(\lambda)$.
(Symmetrically, $f(t)$ modifies $g(t)$ in the same way.) The spectrum of $f + g$ may be more or less "complicated" than the spectrum of f and g alone, and it's an elementary operation in ***both*** the time domain and the frequency domain that produces or eliminates the complications. It's an operation that's also easily undone: See some frequencies you don't like in the spectrum. Then try adding something in or subtracting something out and see what the signal looks like.

We can view the question of using one signal to modify another in either the time domain or in the frequency domain, sometimes with equal ease and sometimes with one point of view preferred. We just looked at sums, what about products? The trivial case is multiplying by a constant, as in $\mathcal{F}(af)(\lambda) = a\mathcal{F}f(\lambda)$. The energies of the harmonics are all affected by the same amount, so, thinking of music for example, the signal sounds the same, only louder or softer. It's much less obvious how to scale the harmonics ***separately***. That is, as a question "in the frequency domain", we ask:

"Is there some combination of the signals $f(t)$ and $g(t)$ so that in the frequency domain the Fourier transform is $\mathcal{F}g(\lambda)\mathcal{F}f(\lambda)$?**"**

In other words, in the time domain can we combine the signal $g(t)$ with the signal $f(t)$ so that the frequency components $\mathcal{F}f(\lambda)$ of $f(t)$ are scaled by the frequency components $\mathcal{F}g(\lambda)$ of $g(t)$? (Once again this is symmetric; we could say that the frequency components $\mathcal{F}g(\lambda)$ are scaled by the frequency components $\mathcal{F}f(\lambda)$.)

The product of the Fourier transforms of $f(t)$ and $g(t)$ is

$$\mathcal{F}g(\lambda)\mathcal{F}f(\lambda) = \int_{-\infty}^{\infty} g(t)\, e^{-i\lambda t} dt \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$

We used different variables of integration in the two integrals because we're going to combine the product into an iterated integral.

$$\int_{-\infty}^{\infty} g(t)\, e^{-i\lambda t} dt \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx = \iint_{-\infty}^{\infty} g(t)f(x)\, e^{-i\lambda t} e^{-i\lambda x} dt dx$$

$$= \iint_{-\infty}^{\infty} g(t)f(x)\, e^{-i\lambda(t+x)} dt dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t) e^{-i\lambda(t+x)} dt \right) f(x) dx.$$

Now make the change of variable $u = t + x$ in the inner integral. Then $t = u - x$, $du = dt$, and the limits are the same. The result is

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t) e^{-i\lambda(t+x)} dt \right) f(x) dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u-x) e^{-i\lambda u} du \right) f(x) dx.$$

Next, switch the order of integration:

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u-x) e^{-i\lambda u} du \right) f(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda u} g(u-x) f(x) du\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda u} g(u-x) f(x) dx\, du = \int_{-\infty}^{\infty} e^{-i\lambda u} \left(\int_{-\infty}^{\infty} g(u-x) f(x) dx \right) du$$

Look at what's happened here. The inner integral is a function of u. Let's set it up on its own:

$$h(u) = \int_{-\infty}^{\infty} g(u-x)f(x)dx.$$

Then the outer integral produces the Fourier transform of h:

$$\int_{-\infty}^{\infty} e^{-i\lambda u}\left(\int_{-\infty}^{\infty} g(u-x)f(x)dx\right)du = \int_{-\infty}^{\infty} e^{-i\lambda u}h(u)du = \mathcal{F}h(\lambda).$$

Switching the variable name for h from $h(u)$ to $h(t)$, we have discovered that the signals $f(t)$ and $g(t)$ are combined into a signal

$$h(t) = \int_{-\infty}^{\infty} g(t-x)\, f(x)dx.$$

In other words,

$$\mathcal{F}h(\lambda) = \mathcal{F}g(\lambda)\mathcal{F}f(\lambda).$$

Now, we give the definition of the ***convolution*** of two functions.

Definition: Suppose f and g are two square-integrable functions. The ***convolution*** of f and g is the function

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)\, g(t)dt. \qquad (C_1)$$

The preceding definition is equivalent to:

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)dt.$$

(The factor $\frac{1}{\sqrt{2\pi}}$ is merely for convenience. If we drop it from the definition of the convolution, it will reappear in its Fourier transform.)

Now we have the following theorem on the Fourier transform of a ***convolution*** of two functions.

Theorem 2.2.20: Suppose f and g are two integrable functions. Then

$$\mathcal{F}[f * g] = \mathcal{F}(f).\mathcal{F}(g) = \hat{f}.\hat{g} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (A_1)$$

$$\mathcal{F}^{-1}[\hat{f}.\hat{g}] = \mathcal{F}^{-1}[\mathcal{F}(f)] * \mathcal{F}^{-1}[\mathcal{F}(g)] = f * g \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (A_2)$$

Proof: To prove Eq. (A_1), we use the definitions of the Fourier transform and convolution:

$$\mathcal{F}[f * g](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f * g\,(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)\, e^{-i\omega x} dx\, g(t) dt.$$

(we write $e^{-i\omega x} = e^{-i\omega(x-t)} e^{-i\omega t}$). After switching the order of integration, we obtain

Letting $u = x - t,\ du = dx$, we obtain

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} du\, e^{-i\omega t} g(t) dt.$$

$$= \mathcal{F}(f)(\omega).\mathcal{F}(g)(\omega) = \hat{f}(\omega)\, \hat{g}(\omega).$$

Eq.(A_2) follows from Eq.(A_1) and the inverse formula for the Fourier transform as follows:

$$\mathcal{F}^{-1}[\hat{f}\hat{g}] = \mathcal{F}^{-1}[\mathcal{F}[f * g]] \qquad (from\ A_1).$$

$$= f * g.$$

Remark 2.2.21: In words: The above result tells us that "**Convolution in the time domain corresponds to multiplication in the frequency domain**".

Remark2.2.22: We also have

$$\mathcal{F}^{-1}(g * f) = \mathcal{F}^{-1} g.\mathcal{F}^{-1} f;\ and\ \mathcal{F}(gf)(\omega) = (\mathcal{F}g * \mathcal{F}f)(\omega).$$

This means that "**Multiplication in the time domain corresponds to convolution in the frequency domain**".

■ **Properties of Convolution**

Convolution behaves in many ways (not all ways) like ***multiplication***. For example,

1. it is commutative: $f * g = g * f$

Proof: We know that $f * g(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)dt = \int_{-\infty}^{\infty} g(x-v)f(v)\, dv$.

(by making the substitution $v = x - t$).

2. It is associative: $(f * g) * h = f * (g * h)$

Proof: (Exercise)

3. It is distributive: $f * (g + h) = f * g + f * h$

Proof: (Exercise)

Example 2.2.23: (**Convolution with the cosine**)

Suppose that f is integrable and even function ($f(x) = f(-x)\ \forall\ x$) and let $g(x) = \cos ax$.

Show that, for all real numbers a, $f * g(x) = \cos ax\ \hat{f}(a)$.

Solution: From the definition and the fact that $f * g = g * f$, we have

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, cos[a(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, [cos\, ax\, cos\, at + sin\, ax\, sin\, at]dt$$

Since f is even, the product $f(t)\, sin\, at$ is odd, hence $\int_{-\infty}^{\infty} f(t)\, sin\, at\, dt = 0$, and so

$$f * g(x) = \cos ax \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, cos\, at\, dt$$

$$= \cos ax \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)[cos\, at - i\, sin\, at]\, dt$$

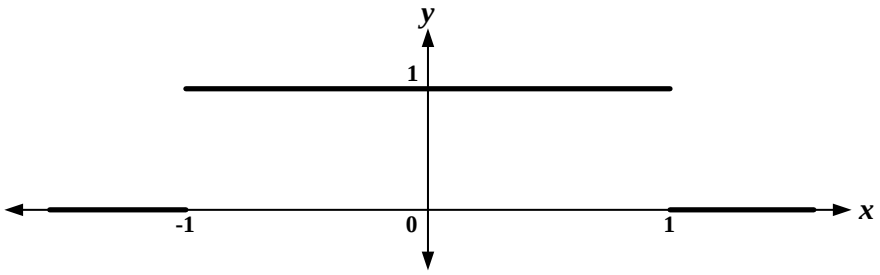
$$= \cos ax \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iat}\, dt = \cos ax\, \hat{f}(a)$$

Example 2.2.24: (**Application to *Theorem 2.2.20***) Fourier Transform of a Convolution.

Consider the function

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & otherwise \end{cases}.$$

The graph of f is shown below.



From ***Example 2.2.1*** we have

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}.$$

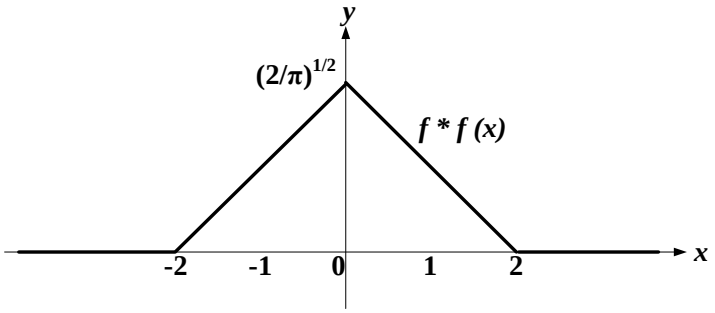
We want to compute $f * f$, the convolution of f with itself. Instead of computing directly from Eq(C_1), we will use ***Theorem 2.2.20*** as follows. We have

$$\mathcal{F}(f * f)(\omega) = \mathcal{F}(f)(\omega).\mathcal{F}(f)(\omega) = \hat{f}(\omega).\hat{f}(\omega) = \left(\hat{f}(\omega)\right)^2 = \frac{2}{\pi} \frac{sin^2(\omega)}{\omega^2}.$$

Using the inverse Fourier transform, with the help of the **table of Fourier transforms**, we find

$$(f * f)(x) = \mathcal{F}^{-1}\left(\frac{2}{\pi} \frac{sin^2(\omega)}{\omega^2}\right) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(1 - \frac{|x|}{2}\right) & if\ |x| < 2 \\ 0 & if\ |x| \geq 2 \end{cases}.$$

The graph of $(f * f)(x)$ is shown in the following **Figure**. Note that $(f * f)(x)$ is **continuous** even though f is **Not**.



Graph of f * f

Table of Fourier Transforms

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{ix\omega}\,d\omega$		$\widehat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
1.	$\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$
2.	$\begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{i\left(e^{-ib\omega} - e^{ia\omega}\right)}{\sqrt{2\,\pi}\omega}$
3.	$\begin{cases} 1 - \frac{\lvert x\rvert}{a} & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ $a > 0$	$2\sqrt{\frac{2}{\pi}}\frac{\sin^2(\frac{a\omega}{2})}{a\,\omega^2}$
4.	$\begin{cases} x & \text{if } \lvert x\rvert < a \\ 0 & \text{if } \lvert x\rvert > a \end{cases}$ $a > 0$	$i\sqrt{\frac{2}{\pi}}\frac{a\omega\cos(a\omega) - \sin(a\omega)}{\omega^2}$
5.	$\begin{cases} \sin x & \text{if } \lvert x\rvert < \pi \\ 0 & \text{if } \lvert x\rvert > \pi \end{cases}$	$i\sqrt{\frac{2}{\pi}}\frac{\sin(\pi\omega)}{\omega^2 - 1}$
6.	$\begin{cases} \sin(ax) & \text{if } \lvert x\rvert < b \\ 0 & \text{if } \lvert x\rvert > b \end{cases}$ $a.b > 0$	$i\sqrt{\frac{2}{\pi}}\frac{\omega\cos(b\omega)\sin(ab) - a\cos(ab)\sin(b\omega)}{\omega^2 - a^2}$
7.	$\frac{1}{a^2 + x^2},\ a > 0$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\omega\rvert}}{a}$
8.	$\frac{x}{a^2 + x^2},\ a > 0$	$-i\sqrt{\frac{\pi}{2}}\,\operatorname{sgn}\omega\, e^{-a\lvert\omega\rvert}$
9.	$\sqrt{\frac{2}{\pi}}\frac{a}{1 + a^2x^2},\ a > 0$	$e^{-\frac{\lvert\omega\rvert}{a}}$
10.	$\frac{\sin ax}{x},\ a > 0$	$\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } \lvert\omega\rvert < a \\ \frac{1}{2}\sqrt{\frac{\pi}{2}} & \text{if } \lvert\omega\rvert = a \\ 0 & \text{if } \lvert\omega\rvert > a \end{cases}$
11.	$\frac{4}{\sqrt{2\pi}}\frac{\sin^2(\frac{1}{2}ax)}{ax^2},\ a > 0$	$\begin{cases} 1 - \frac{\lvert\omega\rvert}{a} & \text{if } \lvert\omega\rvert < a \\ 0 & \text{if } \lvert\omega\rvert > a \end{cases}$
12.	$\frac{4}{\sqrt{2\pi}}\frac{\sin^2(ax) - \sin^2(\frac{1}{2}ax)}{ax^2},\ a > 0$	$\begin{cases} 1 & \text{if } \lvert x\rvert < a \\ (-x + 2a)/a & \text{if } a < x < 2a \\ (x + 2a)/a & \text{if } a < x < 2a \\ 0 & \text{if } \lvert x\rvert > 2a \end{cases}$
13.	$e^{-a\lvert x\rvert},\ a > 0$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2 + \omega^2}$
14.	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases},\ a > 0$	$\frac{1}{\sqrt{2\pi}}\frac{1}{a + i\omega}$
15.	$\begin{cases} 0 & \text{if } x > 0 \\ e^{ax} & \text{if } x < 0 \end{cases},\ a > 0$	$\frac{1}{\sqrt{2\pi}}\frac{1}{a - i\omega}$
16.	$\lvert x\rvert^n e^{-a\lvert x\rvert},\ a > 0,\ n > 0$	$\frac{\Gamma(n+1)}{\sqrt{2\pi}}\left(\frac{1}{(a - i\omega)^{1+n}} + \frac{1}{(a + i\omega)^{1+n}}\right)$

The rest of this section is devoted to studying the ***Gaussian function*** $f(x) = e^{-x^2}$ and its Fourier transform, (see ***Remark 2.2.10***). This function plays a key role in the solution of the heat equation on the line. We need the famous improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \qquad (G_1)$$

Let us give here a more direct proof. We square the integral, use polar coordinates

$(r^2 = x^2 + y^2\ ,\ dxdy = rdrd\emptyset)$, and get

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\, r\, drd\theta = \int_{0}^{2\pi} \left(\frac{-e^{-r^2}}{2}\right)\Big|_0^{\infty} d\emptyset = \frac{1}{2}\int_{0}^{2\pi} d\emptyset = \pi$$

and (G_1) follows upon taking square roots.

Theorem 2.2.25: Let $a > 0$, we have

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right) = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{2a}}\ .$$

This remarkable identity states that $e^{-\frac{x^2}{2}}$ is its own Fourier transform.

Proof: We give an indirect proof based on the operational properties of the Fourier transform. Let $f(x) = e^{-\frac{ax^2}{2}}$. A simple verification shows that f satisfies the first order linear differential equation

$$f'(x) + axf(x) = 0\ .$$

Taking Fourier transform and using ***Theorem 2.2.14*** (properties 1 and 4), we get

$$\omega\hat{f}(\omega) + a\frac{d}{d\omega}\hat{f}(\omega) = 0\ .$$

Thus $\hat{f}$ satisfies a similar first order linear ordinary differential equation.

Solving this equation in $\hat{f}$, we find

$$\hat{f}(\omega) = Ae^{-\frac{\omega^2}{2a}} \ , \quad where\ A\ is\ an\ arbitrary\ constant.$$

To complete the proof, we must show that $A = \frac{1}{\sqrt{a}}$. We have

$$A = \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\, dx.$$

Let $u = \sqrt{\frac{a}{2}}x \ \rightarrow \sqrt{\frac{a}{2}}du = dx$. Thus

$$A = \hat{f}(0) = \frac{1}{\sqrt{a\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{a}} \quad \big(by\ (G_1)\big).$$

Replacing a by $2a$ in ***Theorem 2.2.25***, we get

$$\mathcal{F}\big(e^{-ax^2}\big)(\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}} \ , \qquad a > 0\,.$$

Taking $a = 1$ in ***Theorem 2.2.25***, we obtain

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}\ .$$

Example 2.2.26: (**A special improper integral**)

Writing explicitly the Fourier transform in ***Theorem 2.2.25***, we find that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} (\cos \omega x - i \sin \omega x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} \cos \omega x\, dx = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{2a}}$$

Taking $\omega = 1$ and $a = 2$, we get the interesting identity

$$\int_{-\infty}^{\infty} e^{-x^2} \cos \omega x\, dx = \sqrt{\pi} e^{-\frac{1}{4}}\ . \quad \blacksquare$$

2.3(Application to PDEs)

♦ Transforms of Partial Derivatives

Throughout this section we will suppose that $u(x,t)$ is a function of two variables x and t, where $-\infty < x < \infty$ and $t > 0$. Here, care is needed in identifying the variable with respect to which the Fourier transform is computed. For example, for fixed t, the function $u(x,t)$ becomes a function of the spatial variable x, and as such, we can take its Fourier transform with respect to the x variable. We denote this transform by $\hat{u}(x,t)$. Thus

■ Fourier Transform in the *x* variable:

$$\mathcal{F}\big(u(x,t)\big)(\omega) = \hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x} dx \qquad (1)$$

■ Fourier Transform and Partial Derivatives:

Given $u(x,t)$ with $-\infty < x < \infty$ and $t > 0$, we have

$$\mathcal{F}\left(\frac{\partial}{\partial t} u(x,t)\right)(\omega) = \frac{d}{dt}\hat{u}(\omega,t) \qquad (2)$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial t^n} u(x,t)\right)(\omega) = \frac{d^n}{dt^n}\hat{u}(\omega,t)\,, \qquad n = 1,2,3,\ldots \qquad (3)$$

$$\mathcal{F}\left(\frac{\partial}{\partial x} u(x,t)\right)(\omega) = i\omega\, \hat{u}(\omega,t) \qquad (4)$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial x^n} u(x,t)\right)(\omega) = (i\omega)^n\, \hat{u}(\omega,t)\,, \qquad n = 1,2,3,\ldots \qquad (5)$$

Note that, on the right sides of (2) and (3) we have used ordinary derivatives in t instead of partial derivatives. The reason is to emphasize the crucial fact that, in applying the Fourier transform method, we will transform the partial differential equation in $u(x,t)$ into an ordinary differential equation in $\hat{u}(\omega,t)$, where t is the variable.

Example 2.3.1: (**The wave equation for an infinite string**)

Solve the boundary value problem,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,, \quad -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x) \qquad (initial\ displacement)$$

$$\frac{\partial}{\partial t} u(x,0) = g(x) \qquad (initial\ velocity)$$

Assume that both f and g have Fourier transforms, and give the answer as an inverse Fourier transforms.

Solution: We fix t and take the Fourier transform of both sides of the differential equation and the initial conditions. Using (3) and (5) with $n = 2$, we get

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) = -c^2\omega^2\,\hat{u}(\omega,t) \qquad (6)$$

$$\hat{u}(\omega,0) = \hat{f}(\omega) \qquad (7)$$

$$\frac{d}{dt}\hat{u}(\omega,0) = \hat{g}(\omega) \qquad (8)$$

It is clear that (6) is an ordinary differential equation in $\hat{u}(\omega,t)$, where t is the variable. Let us write (6) in the standard form

$$\frac{d^2}{dt^2}\hat{u}(\omega,t) + c^2\omega^2\,\hat{u}(\omega,t) = 0\,.$$

This is a second order linear differential equation, thus, the general solution of this equation is given by

$$\hat{u}(\omega,t) = A(\omega)\,cos\,c\omega t + B(\omega)\,sin\,c\omega t$$

where $A(\omega)$ and $B(\omega)$ are constants in t. (You should note that while A and B are constants in t, they can depend on ω, which explains writing $A(\omega)$ and $B(\omega)$.)

We determine $A(\omega)$ and $B(\omega)$ from the initial conditions (7) and (8) as follows:

$$\hat{u}(\omega, 0) = A(\omega) = \hat{f}(\omega)$$

$$\frac{d}{dt}\hat{u}(\omega, 0) = c\omega B(\omega) \ = \hat{g}(\omega)$$

So,

$$\hat{u}(\omega, t) = \hat{f}(\omega)\, cos\, c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\, sin\, c\omega t\ .$$

To obtain the solution we use the inverse Fourier transform (**Eq(2) section 2.2**) which is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega \ -\infty < x < \infty$$

and get

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(\omega)\, cos\, c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\, sin\, c\omega t\right] e^{i\omega x}\, d\omega. \qquad (9)\ \blacksquare$$

Formula (9) gives us the solution of the wave boundary value problem in the form of an inverse Fourier transform. That is, in the form of an integral involving the Fourier transform of the wave boundary value problem.

We summarize the Fourier transform method as follows:

Step 1: Fourier transform the given boundary value problem in $u(x, t)$ and get an ordinary differential equation in $\hat{u}(\omega, t)$ in the variable .

Step 2: Solve the ordinary differential equation and find $\hat{u}(\omega, t)$.

Step 3: Inverse Fourier transform $\hat{u}(\omega, t)$ to get $u(x, t)$.

This simple method is successful in treating a variety of partial differential equations, but it has its limitations, since we have to assume that the functions in the problem and its solutions have Fourier transforms. Nevertheless, the method offers us opportunities beyond these limitations, as we now illustrate. Consider (9), which gives us the solution of the wave boundary value problem in the form of an inverse Fourier transform. To simplify the presentation, take $= 0$, then $\hat{g} = 0$ and (9) becomes

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(\omega)\, cos\, c\omega t\right] e^{i\omega x}\, d\omega.$$

Thus, u is the inverse Fourier transform of $\hat{f}(\omega)\, cos\, c\omega t$.

But, $\mathcal{F}^{-1}\left[cos\, a\omega\, \hat{f}(\omega)\right](x) = \frac{f(x-a)+f(x+a)}{2}$ and $\mathcal{F}^{-1}\left[sin\, a\omega\, \hat{f}(\omega)\right](x) = \frac{f(x+a)-f(x-a)}{2i}$

Therefore,

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2}.$$

Example 2.3.2: (**The heat equation for an infinite rod**)

Solve the boundary value problem,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x) \quad (initial\ temperature\ distribution)$$

Assume that f has a Fourier transform, and give the answer as an inverse Fourier transform.

Solution: Fourier transforming the given boundary value problem, we get

$$\frac{d}{dt}\hat{u}(\omega,t) = -c^2\omega^2\, \hat{u}(\omega,t)$$
$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

This is a first order differential equation, thus, the general solution of this equation in t is given by

$$\hat{u}(\omega,t) = A(\omega)e^{-c^2\omega^2 t},$$

where $A(\omega)$ is a constant that depends on ω. Setting $t = 0$ and using the transformed initial condition, we get

$$\hat{u}(\omega,0) = A(\omega) = \hat{f}(\omega)$$

So,

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2\omega^2 t}.$$

To obtain the solution we use the inverse Fourier transform (**Eq(2) section 2.2**) which is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega \ -\infty < x < \infty$$

and get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(\omega) e^{-c^2\omega^2 t}\right] e^{i\omega x}\, d\omega. \qquad (10)\ \blacksquare$$

Formula (10) gives us the solution of the heat boundary value problem in the form of an integral involving the Fourier transform of the initial heat distribution. In the next section, we will compute this integral and give the answer as a convolution of f and a fixed function known as the heat or **Gauss's kernel**. As in the case of the wave equation, you will see that the convolution form of the solution may be applied even if f **does not have a Fourier transform**.

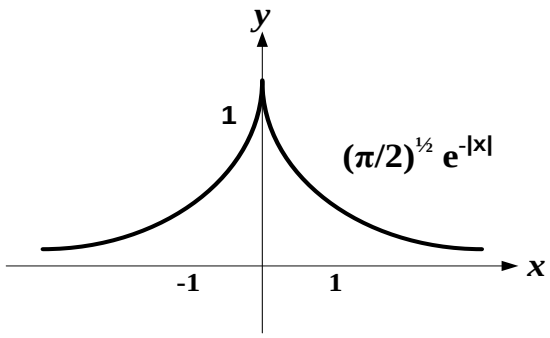
Example 2.3.3: (**The Fourier transform method with mixed derivatives**)

Solve the boundary value problem,

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x^2}\,, \quad -\infty < x < \infty, t > 0\,,$$

$$u(x,0) = \sqrt{\frac{\pi}{2}}\, e^{-|x|}\,.$$

Solution: The function in the initial condition is shown below.



Combining (2) and (4), we obtain

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t \partial x}\right) = \mathcal{F}\left(\frac{\partial}{\partial t}\frac{\partial u}{\partial x}\right) = \frac{d}{dt}\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = i\omega \frac{d\hat{u}}{dt}(\omega, t).$$

Fourier transforming the differential equation and using (5), we get

$$i\omega \frac{d\hat{u}}{dt}(\omega,t) = -\omega^2\, \hat{u}(\omega,t) \quad or \quad \frac{d\hat{u}}{dt}(\omega,t) = i\omega\, \hat{u}(\omega,t)\,.$$

Solving this first order ordinary differential equation, we find

$$\hat{u}(\omega,t) = A(\omega) e^{i\omega t}\,,$$

The initial condition implies that

$$\hat{u}(\omega, 0) = \mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, e^{-|x|}\right) = \frac{1}{1+\omega^2}\,.$$

So,

$$A(\omega) = \frac{1}{1+\omega^2}\,,$$

hence

$$\hat{u}(\omega, t) = \frac{e^{i\omega t}}{1+\omega^2}\,,$$

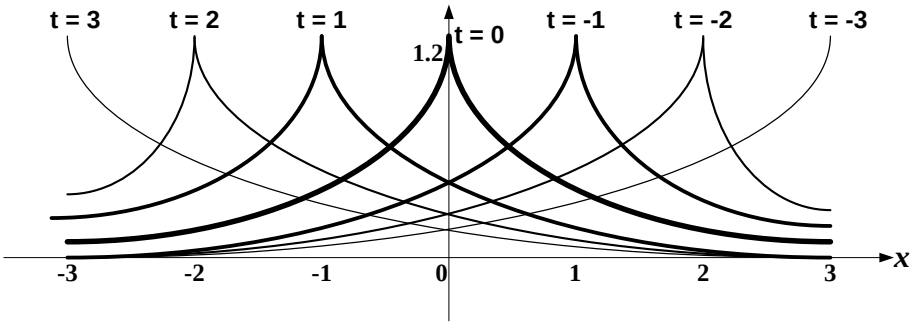
and the solution u is obtained by taking inverse Fourier transform. In this case, we can determine u explicitly by using the ***shifting property*** of the Fourier transform, which is

$$\mathcal{F}\big(f(x-a)\big)(\omega) = e^{-ia\omega}\mathcal{F}(f)(\omega) \qquad (\,shifting\ property).$$

We first note that the inverse Fourier transform of $\frac{1}{1+\omega^2}$ is $\sqrt{\frac{\pi}{2}}\, e^{-|x|}$ (use the table of Fourier transforms). Hence, by the shifting property,

$$u(x,t) = \mathcal{F}^{-1}\left(\frac{e^{i\omega t}}{1+\omega^2}\right) = \sqrt{\frac{\pi}{2}}\, e^{-|x+t|}\,.$$

The solution is illustrated in the following Figure.



Graph of $u(x,t)$ at various values of t.

Our next example deals with a fourth order differential equation that arises in modeling the transverse vibrations of an elastic beam.

Example 2.3.4: (**Transverse vibration of an elastic beam**)
Solve the boundary value problem,

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}, \quad -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x), \qquad \frac{\partial}{\partial t}(x,0) = g(x)$$

where $c > 0$ is a constant. Give your answer in the form of an inverse Fourier transform.

Solution: Fix t and take the Fourier transform of both sides of the given boundary value problem. Using (3) with $n = 2$, we obtain

$$\mathcal{F}(u_{tt}) = \frac{d^2}{dt^2}\hat{u}(\omega, t).$$

Using (5) with $n = 4$, and remembering $i^4 = 1$ we obtain

$$\mathcal{F}(u_{xxxx}) = \omega^4 \, \hat{u}(\omega, t).$$

Thus,

$$\frac{d^2}{dt^2}\hat{u}(\omega, t) + c^2\omega^4 \, \hat{u}(\omega, t) = 0 \quad (-\infty < \omega < \infty),$$

$$\hat{u}(\omega, 0) = \hat{f}(\omega), \qquad \frac{d}{d\omega}\hat{u}(\omega, 0) = \hat{g}(\omega).$$

The second order differential equation in t is similar to the one in ***Example 2.3.1***. Its general solution is

$$\hat{u}(\omega, t) = A(\omega) \cos c\omega^2 t + B(\omega) \sin c\omega^2 t,$$

where $A(\omega)$ and $B(\omega)$ are constants in t. We determine $A(\omega)$ and $B(\omega)$ from the transformed initial conditions as follows:
Setting $t = 0$ and using the transformed initial condition, we get

$$\hat{u}(\omega, 0) = A(\omega) = \hat{f}(\omega)$$

$$\frac{d}{dt}\hat{u}(\omega, 0) = c\omega^2 B(\omega) = \hat{g}(\omega).$$

So,

$$\hat{u}(\omega, t) = \hat{f}(\omega) \cos c\omega^2 t + \frac{1}{c\omega^2}\hat{g}(\omega) \sin c\omega^2 t.$$

To obtain the solution we use the inverse Fourier transform (**Eq(2) section 2.2**) which is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega \quad -\infty < x < \infty$$

and get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(\omega)\, cos\, c\omega^2 t + \frac{1}{c\omega^2} \hat{g}(\omega)\, sin\, c\omega^2 t \right] e^{i\omega x}\, d\omega. \qquad (11)\ \blacksquare$$

So far we have used the Fourier transform method to solve partial differential equations with constant coefficients. As our next example illustrates, the method is also useful in solving problems with nonconstant coefficients.

Example 2.3.5: (**An equation with nonconstant coefficients**)

Solve

$$t\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} = 0 \quad (-\infty < x < \infty\,, \qquad t > 0)\,,$$

$$u(x,0) = f(x)\,.$$

Simplify your answer as much as possible. What are appropriate assumptions on f ?

Solution: The new feature in this example is the presence of the term $t\frac{\partial u}{\partial t}$ in the equation. Since we will use the Fourier transform with respect to x, the variable t will be treated as a constant. Thus

$$\mathcal{F}\left(t\frac{\partial u}{\partial t}\right) = t\,\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = i\omega t\hat{u}(\omega,t).$$

Then

$$i\omega t\hat{u}(\omega,t) + \frac{d}{dt}\hat{u}(\omega,t) = 0,$$

$$\hat{u}(\omega,0) = \hat{f}(\omega)\,.$$

At this point we have to assume that f has a Fourier transform. Solving the first order differential equation in t yields

$$\hat{u}(\omega, t) = A(\omega)e^{-i\frac{t^2}{2}\omega}\,.$$

where the arbitrary constant, $A(\omega)$, is allowed to be a function of ω. Putting $t = 0$ implies

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-i\frac{t^2}{2}\omega}\,.$$

To determine u we use a property of the Fourier transform as we did in ***Example 2.3.3***. This property says that translating a function by a constant a corresponds to multiplying its Fourier transform by $e^{ia\omega}$. Thus

$$u(x, t) = f\left(x - \frac{t^2}{2}\right).$$

Let us check our answer. Assuming that f is differentiable, we have

$$u_x(x, t) = f'\left(x - \frac{t^2}{2}\right);$$

And by the chain rule, $u_t(x, t) = -tf'\left(x - \frac{t^2}{2}\right)$. Thus, $tu_x + u_t = 0$, verifying the equation. At $t = 0$, we get $u(x, 0) = f(x)$, as desired. **<u>The solution that we derived requires only that f be differentiable</u>**.

<u>Exercise 2.3.6</u>: Determine the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \quad -\infty < x < \infty, t > 0$$

$$u(x, 0) = \frac{1}{1 + x^2}\,, \qquad \frac{\partial}{\partial t}u(x, 0) = 0$$

Assume that both f and g have Fourier transforms, and give the answer in the form of an inverse Fourier transform.

<u>Ans</u>: $u(x, t) = \int_{-\infty}^{\infty} e^{-|\omega|}\cos\omega t\, e^{ix\omega}d\omega$.

<u>Exercise 2.3.7</u>: Determine the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = \sqrt{\frac{2}{\pi}} \frac{sin\, x}{x}, \qquad \frac{\partial}{\partial t} u(x,0) = 0.$$

Give your answer in the form of an inverse Fourier transform.

Ans: $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos c\omega t\, e^{ix\omega} d\omega.$

2.4 The Heat Equation and Gauss's Kernel

In the previous section we used the Fourier transform to solve familiar boundary value problems, such as heat and wave problems. The solution was expressed as an inverse Fourier transform involving the Fourier transforms of the initial data. For practical, numerical applications, it is desirable to evaluate the inverse Fourier transform and express the solution in terms of the initial data itself. In this section, we illustrate these ideas with applications.

Let us start with the heat problem (***Example 2.3.2***)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, t > 0, \qquad (1)$$

$$u(x,0) = f(x) \quad (initial\ temperature\ distribution)$$

Assume that f has a Fourier transform. Once we derive the solution, we will apply it in situations where the function f may not have a Fourier transform.
From ***Example 2.3.2***, we know that

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-c^2 \omega^2 t} \qquad (2)$$

and hence, by applying the inverse Fourier transform,

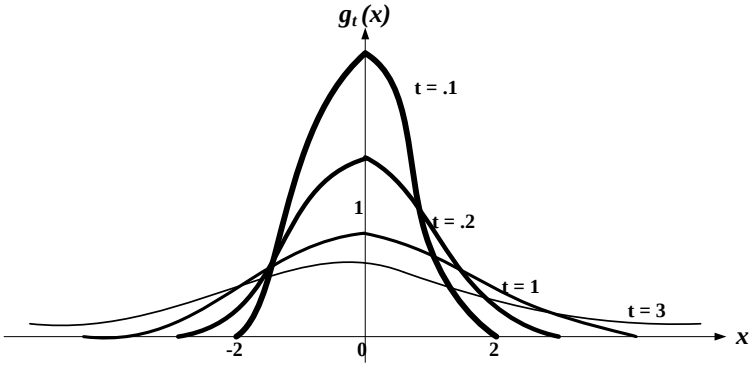
$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\hat{f}(\omega) e^{-c^2 \omega^2 t}\right] e^{i\omega x}\, d\omega. \qquad (3)$$

Our goal in this section is to evaluate this inverse Fourier transform in terms of f.

We note from (2) that $\hat{u}$ is the product of two Fourier transforms, one of them being $\hat{f}$ and the other is $e^{-c^2\omega^2 t}$. Remembering that products of Fourier transforms correspond to convolutions (***Theorem 2.2.20***), we see that u is the convolution of f with the function whose Fourier transform is $e^{-c^2\omega^2 t}$. This function is the so-called **heat kernel** or **Gauss's kernel**

$$g_t(x) = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2 t}}.$$

(we think of the **heat kernel** $g_t(x)$ as a family of functions of x, one function for each $t > 0$, the following **Figure**). This leads us to the following important result.



Theorem 2.4.1: The solution of the heat equation (1) with initial temperature distribution f is the convolution of f with the heat kernel. More explicitly,

$$u(x,t) = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2 t}} * f = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s)\, e^{-\frac{(x-s)^2}{4c^2 t}}\, ds \qquad (4)$$

Proof: We have to show that the function $g_t(x) = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2 t}}$ has Fourier transform $e^{-c^2\omega^2 t}$. But this is immediate from ***Theorem 2.2.25*** by taking ($\frac{a}{2} = \frac{1}{4c^2 t}$) or ($a = \frac{1}{2c^2 t}$) in that theorem, we get

$$\hat{g}_t(\omega) = \frac{1}{c\sqrt{2t}} \sqrt{2c^2 t}\, e^{-\omega^2 c^2 t},$$

as claimed. ■

Note that, ***Theorem 2.2.25*** states that $\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right) = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{2a}}$, $a > 0$.

Remark 2.4.2: A few comments are in order regarding formula (4):

• Because the convolution of functions is commutative, we have
$u(x,t) = f * g_t(x)$ or $u(x,t) = g_t * f(x)$, where f is the initial temperature distribution and g_t is the heat kernel. If you write the convolution explicitely as an integral , you need to introduce a new variable of integration other than x or t. In (4), we used the variable s, but other choices are also possible. You may, for example, write $u(x,t)$ as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-s)\, g_t(s) ds \quad or \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g_t(x-y) dy\,.$$

• Because the heat kernel (as a function of x) decays to 0 very rapidly as $x \to \pm\infty$, formula (4) can be evaluated in many interesting cases even when f does not necessarily have a Fourier transform; for example, when f is bounded or a polynomial.

• Because formula (4) involves dividing by $\sqrt{t}$ and t, we cannot simply plug $t = 0$ into this formula to verify the initial condition $u(x,0) = f(x)$. So, in what sense does the solution (4) satisfy the initial condition ? Let us write the solution (4) as
$u(x,t) = g_t * f(x)$, where g_t is the heat kernel. The solution satisfies the initial condition in the following sense: $\lim_{t\to 0} g_t * f(x) = f(x)$.

If we are interested in numerical values, convolutions are computed very efficiently with the fast Fourier transform (FFT). Also, with the help of (4) we can express the solution in terms of special functions which can be used to study the solution. We illustrates with the so-called **error function** which is defined by

$$erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-z^2}\, dz \quad \forall w \qquad (5)$$

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Example 2.4.3: Determine the solution of the initial-value problem

$$u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty \;,\; t > 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Solution: We take Fourier transform in the x variable. Let $U(\lambda,t)$ be the result. Then, we have

$$u_{tt} \to U_{tt} \quad and \quad u_{xx} \to -\lambda^2 U \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

and the equation becomes

$$U_{tt} + c^2\lambda^2 U = 0 \ , \qquad U(\lambda,0) = \mathcal{F}[f] \quad , \qquad U_t(\lambda,0) = \mathcal{F}[g] \dots\dots\dots\dots\dots (4)$$

The general solution is

$$U(\lambda,t) = A(\lambda)\cos c\lambda t + \ B(\lambda)\sin c\lambda t \dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

To fix A and B, we use the initial values:

$$\mathcal{F}[f](\lambda) = U(\lambda,0) = A(\lambda) \dots\dots\dots\dots\dots (6)$$

$$\mathcal{F}[g](\lambda) = U_t(\lambda,0) = c\ \lambda\ B(\lambda) \dots\dots\dots\dots\dots (7)$$

Therefore,

$$U(\lambda,t) = \mathcal{F}[f](\lambda)\cos c\lambda t + \ \mathcal{F}[g](\lambda)\frac{\sin c\lambda t}{c\lambda} \dots\dots\dots\dots\dots\dots (8)$$

Taking the inverse transform we obtain

$$u(x,t) = f * \mathcal{F}^{-1}[\cos c\lambda t] + g * \mathcal{F}^{-1}\left[\frac{\sin c\lambda t}{c\lambda}\right] \dots\dots\dots\dots\dots\dots (9)$$

To obtain the formula in variables x; t we need to compute

$$\mathcal{F}^{-1}[\cos c\lambda t] \ \ and \ \ \mathcal{F}^{-1}\left[\frac{\sin c\lambda t}{c\lambda}\right] \dots\dots\dots\dots\dots\dots\dots (10)$$

It is easy to see that direct integration using the inverse transform formula does not quite work. We need to compute them indirectly.

Since $\cos c\lambda t = \frac{1}{2}\left(e^{ic\lambda t} + e^{-ic\lambda t}\right) \dots\dots\dots\dots\dots (11)$, we have

$$\mathcal{F}^{-1}[\cos c\lambda t](x) = \frac{1}{2}\left[\mathcal{F}^{-1}\left(e^{ic\lambda t}\right) + \mathcal{F}^{-1}\left(e^{-ic\lambda t}\right)\right]$$

$$= \frac{1}{2}\left[\mathcal{F}^{-1}(1)(x+ct) + \mathcal{F}^{-1}(1)(x-ct)\right] \dots\dots\dots\dots (12)$$

To find out $\mathcal{F}^{-1}(1)$, we use the property of ***convolution***. We have

$$\mathcal{F}^{-1}(1) * f = \mathcal{F}^{-1}(\mathcal{F}\left(\mathcal{F}^{-1}(1)\mathcal{F}(f)\right) = \mathcal{F}^{-1}(\mathcal{F}) = f \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Therefore,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}^{-1}(1)(x-y)\,f(y)dy = f(x)\ \ \forall x \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

This tells us that

$$\mathcal{F}^{-1}(1)(x) = \sqrt{2\pi}\,\delta(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

As a consequence

$$\mathcal{F}^{-1}[\cos c\lambda t] = \sqrt{\frac{\pi}{2}}[\delta(x+ct) + \delta(x-ct)] \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

For the second term, recall that we have shown that

$$\mathcal{F}[f_a](\lambda) = \sqrt{\frac{2}{\pi}}\frac{\sin a\lambda}{\lambda} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

where $f_a = 1\ for\ |x| < a\ \ and\ \ 0\ \ \ outside\ .As\ a\ consequence$

$$\mathcal{F}^{-1}\left[\frac{\sin c\lambda t}{c\lambda}\right] = \frac{1}{c}\mathcal{F}^{-1}\left[\frac{\sin c\lambda t}{\lambda}\right] = \sqrt{\frac{\pi}{2}}\frac{1}{c}f_{ct} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Finally we compute

$$u(x,t) = f * \mathcal{F}^{-1}[\cos c\lambda t] + g * \mathcal{F}^{-1}\left[\frac{\sin c\lambda t}{c\lambda}\right]$$

$$= \sqrt{\frac{\pi}{2}}[f * \delta(x+ct) + f * \delta(x-ct)] + \sqrt{\frac{\pi}{2}}\frac{1}{c}g * f_{ct}$$

$$= \sqrt{\frac{\pi}{2}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\delta(x+ct-y)dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\delta(x-ct-y)dy\right]$$
$$+ \sqrt{\frac{\pi}{2}}\frac{1}{c}\int_{-\infty}^{\infty}g(y)f_{ct}\delta(x-y)dy$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\,dy \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

2-D Continuous Fourier Transform (CFT)

The 2-D CFT is given by

$$\mathcal{F}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)} dxdy \quad ; j = \sqrt{-1} \qquad (1)$$

and the 2-D inverse CFT (ICFT) is given by

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(u,v)\, e^{j2\pi(ux+vy)} dudv \qquad (2)$$

where $f(x,y)$ be a given continuous function at *x* and *y* ; and $\mathcal{F}(u,v)$ be a continuous Fourier transform of $f(x,y)$ and *u*,*v* are the frequency variables.

The Fourier transform is generally complex; by using Euler's formula we can express FT in complex form:

$$\mathcal{F}(\omega) = R(\omega) + j\, I(\omega) = |\mathcal{F}(\omega)| e^{j\theta(\omega)} \qquad (3)$$

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)\, [cos(2\pi\omega x) - j\, sin(2\pi\omega x)] dx \qquad (4)$$

and the inverse in complex form is:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}(\omega)\, [cos(2\pi\omega x) + j\, sin(2\pi\omega x)] d\omega \qquad (5)$$

where $R(\omega)$ and $I(\omega)$ are the real and imaginary components of $\mathcal{F}(\omega)$, respectively, and The (face angle) $\theta(\omega)$ is given by:

$$\theta(\omega) = tan^{-1}(I(\omega)/R(\omega)) \qquad (6)$$

$$|\mathcal{F}(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)} \qquad (7)$$

where $|\mathcal{F}(\omega)|$ is called the Fourier spectrum of *f*(x) (the magnitude of $\mathcal{F}(\omega)$).

Discrete Fourier Transform (DFT)

If the input to the DFT is a sequence of numbers rather than a continuous function of time *f(x)* then we can use discrete Fourier transform. The sequence of numbers usually results from periodically sampling a continuous function *f(x)* at an interval of *T*. The DFT is derived from a continuous-time function *f(x)* using *N* samples taken at times $x = T, 2T, \dots, (N-1)T$ where $T = \Delta x$ is the sampling interval. These N samples of *f(x)* form a data sequence:

$$\{f(x_0), f(x_0 + T), f(x_0 + 2T), \dots, f(x_0 + (N-1)T)\}$$

Therefore, $f(x) = f(x_0 + kT)$ where k now assumes the discrete values 0,1,2,….*N*-1. **Figure 1** illustrates this concept.

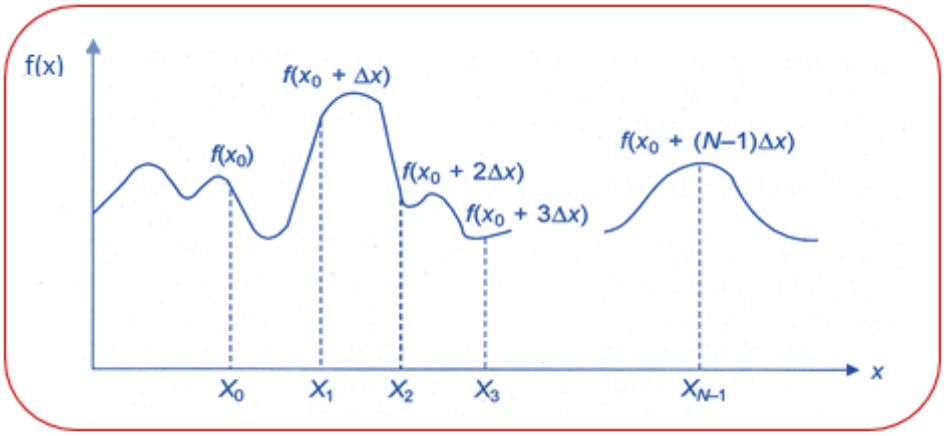


Figure 1: Sampling a Continuous Function

The discrete Fourier transform is the equivalent to the CFT for signals (functions) defined only at *N* sample times (i.e. a finite sequence of data). Let *N* samples denoted by $f[0], f[1], \dots, f[N-1]$.

The 1-D DFT is given by:

$$F[n] = \frac{1}{N} \sum_{k=0}^{N-1} f[k]\, e^{\frac{-2j\pi nk}{N}}, n = 0,1, \dots, N-1, j = \sqrt{-1} \qquad (8)$$

The inverse of 1-D DFT is given by:

$$f[k] = \sum_{n=0}^{N-1} F(n)\, e^{\frac{2j\pi nk}{N}}, k = 0,1, \dots, N-1 \qquad (9)$$

where $f[k]$ is a function of one variable, $k = 0,1,\dots,N-1$ and $F[n]$ is the DFT of $f[k]$, n is the frequency variables, $n = 0,1,\dots,N-1$.
The values $n = 0,1,\dots,N-1$ in the DFT given in Eq.(8) correspond to samples of the continuous transform at values $0, \Delta n, 2\Delta n, \dots, (N-1)\Delta n$.

Example: Figure (2a) shows four samples of a continuous function, $f(t)$, taken ΔT units apart. Let a function be sampled as $\{f(0)=1, f(1)=2, f(2)=4, f(3)=4\}$ where the value of t are $0, 1, 2$ *and* 3. We could be referring to any four samples as follows:

Applying Eq.(8), In this case, $N = 4$. When $u = 0$

$$F[n] = \frac{1}{N}\sum_{k=0}^{N-1} f[k]\, e^{\frac{-2j\pi nk}{N}}$$

$$F[0] = \frac{1}{4}\sum_{k=0}^{3} f[k]\, e^{0} = \frac{1}{4}\big[f[0] + f[1] + f[2] + f[3]\big] = \frac{1}{4}[1+2+4+4] = \frac{11}{4}$$

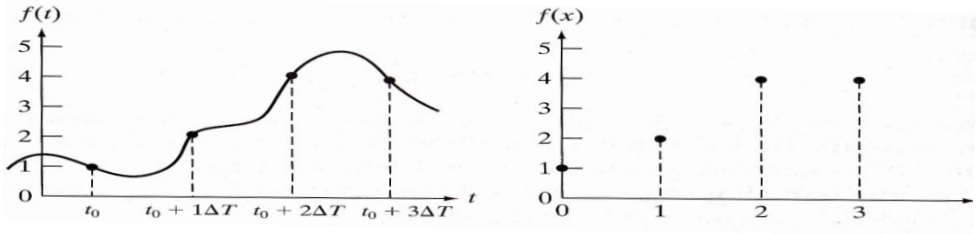
Similarly we can be computing $F(1), F(2)$ *and* $F(3)$, where :

$$F(1) = \frac{-3+2j}{4}, F(2) = \frac{-(1+0j)}{4} \text{ and } F(3) = \frac{-(3+2j)}{4}$$

Figure (2) shows the sampled value in the *k*-domain. The inverse transform can be computing applying Equation (9).

$$f[k] = \sum_{n=0}^{N-1} F(n)\, e^{\frac{2j\pi nk}{N}}$$

$$f[0] = \textstyle\sum_{n=0}^{3} F(n)\, e^{\frac{2j\pi n0}{N}} = \frac{1}{4}[11 - 3 + 2j - 1 - 3 - 2j] = \frac{1}{4}[4] = 1$$



(a)A function and t, is a continuous (b) samples in the *x*-domain

Figure (2): Continuous Variable and Samples in the *x*-domain

Example: Calculate the four-point DFT of the aperiodic sequence $x[k]$ of length $N = 4$, which is defined as follows:

$$x[k] = \begin{cases} 2 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 1 & k = 3 \end{cases}$$

Solution : By using Eq. (8), the four-point of DFT is

$$X[n] = \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-j(\frac{2\pi nk}{4})}$$

$$= \frac{1}{4}\left(2 + 3.\ e^{-j(\frac{2\pi n}{4})} - 1.\ e^{-j\left(\frac{2\pi(2)n}{4}\right)} + 1.\ e^{-j\left(\frac{2\pi(3)n}{4}\right)}\right)$$

On substituting different values of n, we obtain

$$n = 0 \qquad X[0] = \frac{1}{4}(2 + 3 - 1 + 1) = \frac{5}{4};$$

$$n = 1 \qquad X[1] = \frac{1}{4}\left(2 + 3\,e^{-j\left(\frac{2\pi}{4}\right)} - 1\,e^{-j\left(\frac{2\pi(2)}{4}\right)} + 1\,e^{-j\left(\frac{2\pi(3)}{4}\right)}\right)$$

$$= \frac{1}{4}(2 + 3(-j) - 1(-1) + 1(j)) = \frac{3 - 2j}{4}$$

$$n = 2 \quad X[2] = \frac{1}{4}\left(2 + 3\,e^{-j\left(\frac{2\pi(2)}{4}\right)} - 1\,e^{-j\left(\frac{2\pi(2)(2)}{4}\right)} + 1\,e^{-j\left(\frac{2\pi(3)(2)}{4}\right)}\right)$$

$$= \frac{1}{4}\big(2 + 3(-1) - 1(1) + 1(-1)\big) = \frac{-3}{4}$$

$$n = 3 \qquad X[3] = \frac{1}{4}\left(2 + 3\,e^{-j\left(\frac{2\pi(3)}{4}\right)} - 1\,e^{-j\left(\frac{2\pi(2)(3)}{4}\right)} + 1\,e^{-j\left(\frac{2\pi(3)(3)}{4}\right)}\right)$$

$$= \frac{1}{4}\big(2 + 3(j) - 1(-1) + 1(-j)\big) = \frac{3 + 2j}{4}$$

Example: Evaluate inverse of DFT of

$$F[n] = \begin{cases} \frac{5}{4} & n = 0 \\ \frac{3 - 2j}{4} & n = 1 \\ \frac{-3}{4} & n = 2 \\ \frac{3 + 2j}{4} & n = 3 \end{cases}$$

Solution : By using Eq. (9),

$$f[k] = \sum_{k=0}^{3} F[n] e^{j\left(\frac{2\pi nk}{4}\right)}$$

$$= [\frac{5}{4} + \frac{3-2j}{4} e^{j\left(\frac{2\pi k}{4}\right)} - \frac{3}{4} e^{j\left(\frac{2\pi(2)k}{4}\right)} + \frac{3+2j}{4} e^{j\left(\frac{2\pi(3)k}{4}\right)}]$$

On substituting different values of *k*, we obtain

$$f[0] = \frac{1}{4}[5 + (3-2j) - 3 + (3+2j)] = 2\ ;$$

$$f[1] = \frac{1}{4}\left[5 + (3-2j)e^{j\left(\frac{2\pi}{4}\right)} - 3e^{j\left(\frac{2\pi(2)}{4}\right)} + (3+2j)e^{j\left(\frac{2\pi(3)}{4}\right)}\right]$$

$$= \frac{1}{4}[5 + (3-2j)(j) - 3(-1) + (3+2j)(-j)] = 3;$$

$$f[2] = \frac{1}{4}\left[5 + (3-2j)e^{j\left(\frac{2\pi(2)}{4}\right)} - 3e^{j\left(\frac{2\pi(2)(2)}{4}\right)} + (3+2j)e^{j\left(\frac{2\pi(3)(2)}{4}\right)}\right]$$

$$= \frac{1}{4}[5 + (3-2j)(-1) - 3(1) + (3+2j)(-1)] = -1;$$

$$f[3] = \frac{1}{4}\left[5 + (3-2j)e^{j\left(\frac{2\pi(3)}{4}\right)} - 3e^{j\left(\frac{2\pi(2)(3)}{4}\right)} + (3+2j)e^{j\left(\frac{2\pi(3)(3)}{4}\right)}\right]$$

$$= \frac{1}{4}[5 + (3-2j)(-j) - 3(-1) + (3+2j)(j)] = 1$$

<u>Note</u>: Eq. (8) can be written in matrix form as

$$\underbrace{\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ \cdot \\ \cdot \\ \cdot \\ F[N-1] \end{bmatrix}}_{DFT\ vector:\vec{F}} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{N-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & W^{N-1} & W^{N-2} & \cdots & W \end{bmatrix}}_{DFT\ matrix\ F} \underbrace{\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \cdot \\ \cdot \\ \cdot \\ f[N-1] \end{bmatrix}}_{Signal\ vector:\vec{f}} \qquad (10)$$

where $W = e^{-j(2\pi/N)}$ and $W = W^{2N}$

Similarly, for the inverse DFT can be written as

$$\underbrace{\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \cdot \\ \cdot \\ \cdot \\ f[N-1] \end{bmatrix}}_{Signal\ vector:\vec{f}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & \dots & W^{N-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & W^{N-1} & W^{N-2} & \cdots & W \end{bmatrix}}_{DFT\ matrix\ G=F^{-1}} \underbrace{\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ \cdot \\ \cdot \\ \cdot \\ F[N-1] \end{bmatrix}}_{DFT\ vector\ \vec{F}:} \qquad (11)$$

where $W = e^{j(2\pi/N)}$

The 2-D DFT equation given by

$$F(u,v) = \frac{1}{NM}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)e^{-2j\pi(\frac{ux}{N}+\frac{vy}{M})} \quad u = 0,1,\dots,N-1, v = 0,1,\dots,M-1 \quad (12)$$

Inverse of 2-D DFT defined by

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} F(u,v)e^{2j\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}, \quad x = 0,1,\dots,N-1, y = 0,1,\dots,M-1 \qquad (13)$$

Where $f(x,y)$ is a function of two variables *x* and *y*, and $F(u,v)$ is a DFT of $f(x,y)$.

The following are some most important properties for DFT:

■ Separability

One important property of the DFT is its ***separability*** in which the two-dimensional basis function can be decomposed into two product terms. If the basis functions are separable, then the result can be found by successive application of two, one-dimensional transforms. Figure 3 illustrates this property.

Proof: Since the forward (DFT) is

$$F(u,v) = \frac{1}{NM}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)e^{-2j\pi(\frac{ux}{N}+\frac{vy}{M})} \quad u = 0,1,\dots,N-1, v = 0,1,\dots,M-1$$

and the backward (DFT) is

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} F(u,v)e^{2j\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}, \quad x = 0,1,\dots,N-1, y = 0,1,\dots,M-1$$

If $f(x,y)$ is a square matrix, that is $M = N$.

Since $f(x,y)$ is NxN then the kernel is separable

$e^{-2j\pi(\frac{ux+vy}{N})} = e^{-2j\pi(\frac{ux}{N})} e^{-2j\pi(\frac{vy}{N})}$

Rewrite $F(u,v)$ as follows

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} e^{-2j\pi(\frac{ux}{N})} \left\{ \sum_{y=0}^{N-1} f(x,y)\, e^{-2j\pi(\frac{vy}{N})} \right\}$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} e^{-2j\pi(\frac{ux}{N})} . \frac{1}{N} \sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{vy}{N})}$$

Since $\sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{vy}{N})} = F[x,v]$, we can compute $F[x,v]$ by N multiply by DFT of rows of $f(x,y)$ as:

$$\sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{vy}{N})} = F[x,v] = N \left\{ \frac{1}{N} \sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{vy}{N})} \right\}$$

Then, we calculate $F[u,v]$ by DFT of columns of $F[x,v]$

$$F[u,v] = \frac{1}{N} \sum_{x=0}^{N-1} e^{-2j\pi(\frac{ux}{N})} \times N \times F[x,v] = \sum_{x=0}^{N-1} e^{-2j\pi(\frac{ux}{N})} \times F[x,v]$$

■ **Symmetry**

if $f(x,y)$ is real function, then DFT of $f(x,y)$ is a conjugate symmetric about the origin and its magnitude is a symmetric about the origin.

$F[u,v] = F^*[-u,-v] \rightarrow |F[u,v]| = |F^*[-u,-v]|, where\ *\ denotes\ the\ conjugate.$

<u>Proof</u>: We know that

$$F[u,v] = \frac{1}{NM} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{ux}{M} + \frac{vy}{N})}$$

$$F[-u,-v] = \frac{1}{NM} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2j\pi(\frac{-ux}{M} - \frac{vy}{N})} = \frac{1}{NM} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{2j\pi(\frac{ux}{M} + \frac{vy}{N})}$$

$$F^*[-u,-v] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Thus, the DFT of $f(x,y)$ is conjugate symmetric about the origin.

Since magnitude $|F(\omega)| = \sqrt{R^2(\omega)+I^2(\omega)}$, and since

$F^*[-u,-v] = F[u,v] \rightarrow |F[u,v]| = |F[-u,-v]|$

Thus, the magnitude of DFT of $f(x,y)$; $|F[u,v]|$ is symmetric about the origin.

■ Periodicity

The DFT and its inverse are infinitely periodic in both the *u*,*v* and *x*,*y* directions, respectively, as

$F[u,v] = F[u+M,v] = F[u,v+N] = F[u+M,v+N]$

$f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$

Proof: Since

$$F[u,v] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$F[u+M,v] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(u+M)x}{M}+\frac{vy}{N})}$$

$$F[u+M,v] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(ux)}{M})}\, e^{-2j\pi(\frac{(Mx)}{M})}\, e^{-2j\pi(\frac{(vy)}{N})}$$

$$F[u+M,v] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(ux)}{M}+\frac{(vy)}{N})}\, e^{-2j\pi(\frac{(Mx)}{M})} = \; F[u,v].$$

Since $e^{-2j\pi(\frac{(Mx)}{M})} = 1 \; \forall x = 0,1,2,\dots,M-1.$

Now since

$$F[u,v+N] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{ux}{M}+\frac{(v+N)y}{N})}$$

$$F[u, v+N] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(ux)}{M})}\, e^{-2j\pi(\frac{(vy)}{N})}e^{-2j\pi(\frac{(Ny)}{N})} = \; F[u,v]$$

Since $e^{-2j\pi(\frac{(Ny)}{N})} = 1\ \forall x = 0,1,2,\dots,N-1.$

On the other hand,

$$F[u+M, v+N] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(u+M)x}{M}+\frac{(v+N)y}{N})}\, e^{-2j\pi(\frac{(Ny)}{N})}$$

$$F[u+M, v+N] = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2j\pi(\frac{(ux)}{M})}\, e^{-2j\pi x}e^{-2j\pi(\frac{(vy)}{N})}e^{-2j\pi y} = \; F[u,v].$$

Therefore

$$F[u,v] = F[u+M,v] = F[u,v+N] = F[u+M,v+N].$$

To prove the periodicity of the backward relationship we must followed the similar steps.

Remark:

DFT has an important tool in signal processing because that it implemented using fast Fourier transforms (FFT). The FFT is an algorithm to make computation of the DFT quick and efficient. It is not a unique type of Fourier transform ob its own.

The problem with the DFT is that it is not an optimal transformation, since it does not diagonalise the covariance matrix. In addition, the inverse DFT generates samples which are periodic extension of the first N samples, that is,

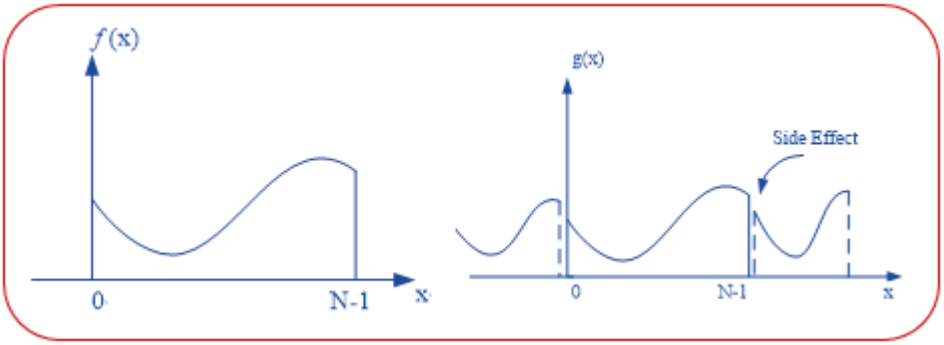
$$g(x) = f(x), \quad x = 0,1,2,\dots,N-1, \qquad (14)$$

and

$$g(x) \neq f(x), \quad x = N, N+1, N+2, \dots\dots\dots, \qquad (15)$$

$$g(x+N) = g(x) \qquad (16)$$

This periodicity in *DFT* causes discontinuities at the beginning and end of each block. This effect can be seen in **Figure 4** .



(a) **(b)**

Figure 4: Side Effect in DFT

A big problem in using Fourier transform is that it is **not localized**. It does not reveal how the signal's frequency contents vary with time. This localization is of great importance in several applications, and in particular in the problem of function representation. The existence of a certain frequency detects the presence of some object, and the localization of a frequency allows us to determine the object position. Because the temporal structure of the signal is not detected, the feature of the Fourier transform is limited; specifically, it is not suitable for analyzing non-stationary signals. In order to overcome the limitation of the Fourier transform, the Short-Time Fourier Transform (STFT) was introduced by Dennis Gabor in 1946. The STFT is a modification of in the definition of the Fourier transform to obtain a transform with better localization properties in the time-frequency domain. It consists of an analysis window of specific length that glides during the signal along the time axis to perform a "time-localized" Fourier transform.

Short Time Fourier Transform (STFT)

The Short-Time Fourier Transform (STFT) or windowed Fourier transform, is the simplest time frequency representation. Instead of transforming the whole signal, all at once using the Fourier transform, it is transformed on a (block-by-block) basis using a moving time-window, centered at the time instant τ.

The STFT employs a sliding window function *g(t)* that is centered at time τ. For each certain τ, a time-localized Fourier transform is performed on the signal *x(t)* in the window. If the window shape is given by *g*(*t*), the *STFT* is formally given by

$$F(\omega,\tau) = \int_{-\infty}^{\infty} x(t)g(t-\tau)e^{-j\omega t}\,dt \qquad (17)$$

This means that the signal (function) $x(t)$ which is to be analyzed is multiplied with an analysis window $g(t-\tau)$ which is centered at time τ **.** Then, the window is movable by τ along the time line, and another Fourier transform is performed. Through such sequential operations, Fourier transform of the whole signal can be performed. The signal segment in the window function is assumed to be approximately stationary. As a result, the Short-Time Fourier Transform (STFT) decomposes a time domain signal into a 2D time-frequency representation, and alteration of the frequency content of that signal within the window function are revealed, as illustrated in **Figure 5**.

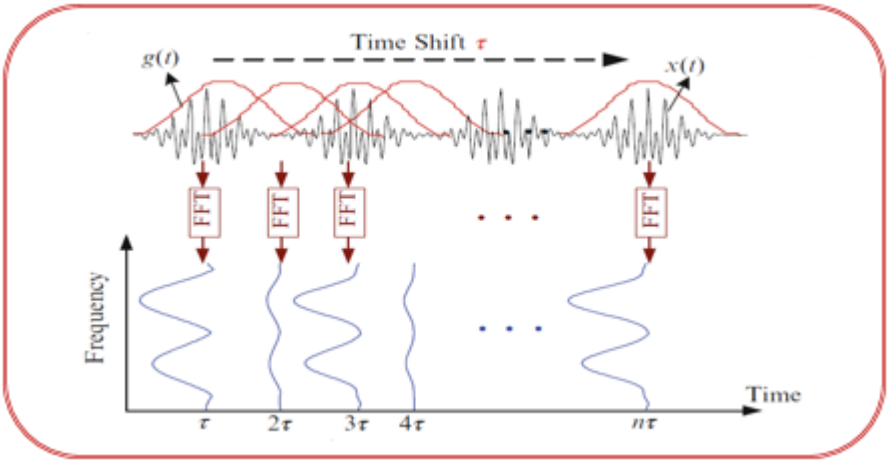


Figure 5: Illustration of Short-Time Fourier Transform on the Test Signal *x*(*t*)

Discrete Cosine Transform (*DCT*)

The Discrete Cosine Transform (DCT) is a member of a family of sinusoidal unitary transforms. It is one of the most efficient transform coding schemes. It was introduced to produce and solve the problem of discontinuity at the end of the data blocks inherit with the DFT.

This transform is an orthogonal, separable, and real transform which translates the image information from spatial domain to frequency domain to be represented in a more compact form. It is very closely related to a DFT. Being separable means that the multidimensional transform can be decomposed into successive application of one-dimensional transforms in the appropriate directions. The DCT is the basis of many image compression methods. For example, the standard JPEG (Joint Photographers Expert Group), for which DCT is carried out in 8x8 image blocks (matrices) existed as the main image compression standard.

It is a fast transform and has excellent energy compaction for highly correlated data. These functions illustrate the power of mathematics in the field of image compression. Its kernel was described a set of **complete**, **orthogonal** discrete cosine basis functions and is an **invertible linear** transform.

The equation for the one-dimensional DCT of a sequence $\{f(x), x = 0, 1, \ldots, N-1\}$ that is zero outside the interval $0 \leq x \leq N\text{-}1$ is defined as:

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{(2x+1)u\pi}{2N}\right], \qquad u = 0,1,\dots,N-1 \qquad (18)$$

The inverse transformation is given by

$$f(x) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{(2x+1)u\pi}{2N}\right], \qquad x = 0,1,\dots,N-1 \qquad (19)$$

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{2}} & for \quad u = 0 \\ 1 & for \quad u \neq 0 \end{cases} \qquad (20)$$

The one-dimensional *DCT* basis vectors can be obtained from

$$C_{u,x} = \sqrt{\frac{2}{N}} \alpha(u) cos\left(\frac{(2x+1)u\pi}{2N}\right), x = 0,1,\dots,N-1\,; u = 0,1,\dots,N-1 \quad (21)$$

Since the *DCT* is a separable transform, the two-dimensional DCT and its inverse can be obtained in two steps by successive applications of the one-dimensional DCT and its inverse. For image processing applications, the two-dimensional DCT of an $N \times N$ block of pixels, *f(x, y)* can be defined as:

$$C(u,v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\left[\frac{(2x+1)u\pi}{2N}\right] cos\left[\frac{(2y+1)v\pi}{2N}\right], \qquad (22)$$

where, *f(x, y)* denotes a two dimensional sequence of *N* x *N* points, *C(u, v)* denotes $N \times N$ points *DCT* of the block *f(x, y)* and *x , y = 0,1, …,N-1*; *u ,v = 0,1, …, N-1*.

The inverse *DCT* is defined as

$$f(x,y) = \alpha(u)\alpha(v) \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} C(u,v) \cos\left[\frac{(2x+1)u\pi}{2N}\right] cos\left[\frac{(2y+1)v\pi}{2N}\right], \qquad (23)$$

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & for \quad u = 0 \\ \sqrt{\frac{2}{N}} & for \quad u = 1,2,\dots,N-1 \end{cases} \qquad (20)$$

The transform matrix of the two-dimensional *DCT* is given by

$$C_{u,v,x,y} = \sqrt{\frac{2}{N}} \alpha(u)\alpha(v) cos\left(\frac{(2x+1)u\pi}{2N}\right) cos\left(\frac{(2x+1)v\pi}{2N}\right), x,y,u,v = 0,1,\dots,N-1 \quad (21)$$

The coefficient of the *DCT* matrix with zero frequency (*C* (*0, 0*)) is called the "DC coefficient," while the remaining coefficients are called the "AC coefficients", which reflect variations in grey level values in certain direction at certain rate as shown in **Figure** 7.

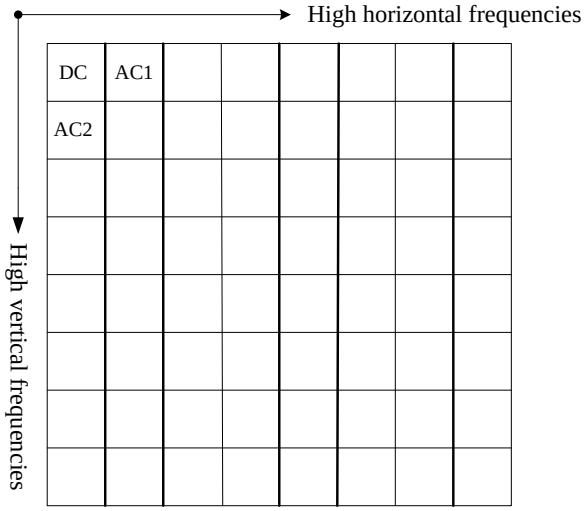


Figure 7: The DCT block structure.

Conveniently, in most image subblocks the energy is concentrated in the DC and low AC frequency bands. That is, the image information (energy) is usually concentrated in the low frequency region (the top left corner). The high frequency region is located in the bottom right corner.This allows for the suppression of the higher AC frequency bands thus allowing the block to be represented by fewer coefficients in its transformed state than when it existed in its original state. One simple observation is that each *DCT* coefficient *C (u, v)* is a linear combination of all pixel values within the block.

DCT has excellent decorrelation property in which the basis vectors used to represent the original signal or image block are signal or image *independent*. These linear basis functions can be seen in **Figure 8** for an 8×8 image block.

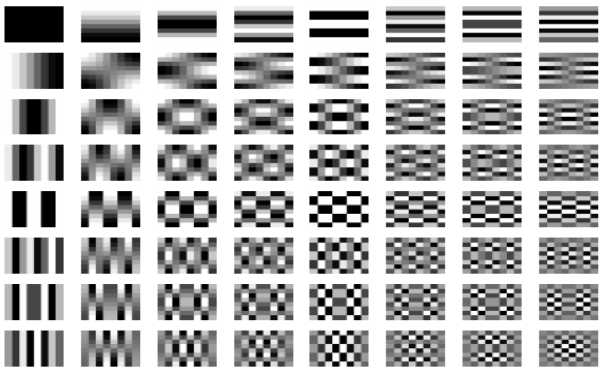


Figure 8: Basis functions of an 8x8 DCT.

The Laplace Transform: Basic Definitions and Results

The Laplace transform is one of many "integral transforms" in general use. It is a mathematical tool based on integration that has a number of applications. In particular, it can simplify the solving of many differential equations. The process of solution consists of three main steps:

- The given "hard" problem is transformed into a "simple" equation.
- This simple equation is solved by purely algebraic manipulations.
- The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration. The above procedure can be summarized by Figure 1.

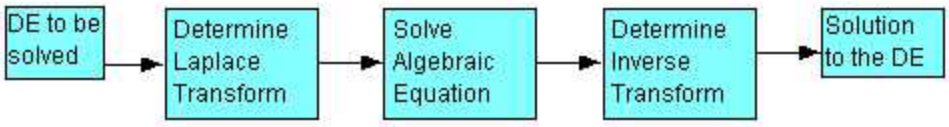


Fig.1

We will find it particularly useful when dealing with nonhomogeneous equations in which the forcing functions are not continuous. This makes it a valuable tool for engineers and scientists dealing with "real-world" applications.

Basic Definition and Examples:

Let f(t) be defined for $t \geq 0$. Then the Laplace transform of f; which is denoted by $L\,[f(t)]$ or by $F(s)$, is defined by the following equation

$$L[f(t)] = \lim_{T\to\infty} \int_0^T f(t)\, e^{-st} dt = \int_0^\infty f(t)\, e^{-st} dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The integral which defined a Laplace transform is an *improper integral*. An *improper integral* may converge or diverge, depending on the integrand. When the improper integral is convergent then we say that the function $f(t)$ possesses a *Laplace transform*. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral?

Ex.1: Find the Laplace transform, if it exists, of each of the following functions

$a)\ f(t) = e^{at}$, $b)\ f(t) = 1$, $c)\ f(t) = t$, $d)\ f(t) = e^{t^2}$.

Solution: **(*a*)** Using the definition of Laplace transform we see that

$$L[e^{at}] = \int_0^{\infty} e^{at}\, e^{-st} dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \lim_{T\to\infty} \int_0^{T} e^{-(s-a)t}\, dt.$$

But;

$$\int_0^{T} e^{-(s-a)t}\, dt = \begin{cases} T & if\ s = a \\ \dfrac{1 - e^{-(s-a)T}}{s-a} & if\ s \neq a \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$L[e^{at}] = F(s) = \frac{1}{s-a}\ ; s > a\ .$$

(*b*) In a similar way to what was done in part (*a*), we find

$$L[1] = \int_0^{\infty} e^{-st}\, dt = \lim_{T\to\infty} \int_0^{T} e^{-st}\, dt = \frac{1}{s}\ ; \quad s > 0\ .$$

(*c*) We have

$$L[t] = \int_0^{\infty} t\, e^{-st}\, dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^{\infty} = \frac{1}{s^2}\ ; \quad s > 0.$$

(*d*) Again using the definition of Laplace transform we find

$$L\left[e^{t^2}\right] = \int_0^{\infty} e^{t^2}\, e^{-st}\, dt = \int_0^{\infty} e^{t^2 - st} dt$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \geq \int_0^\infty dt$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (**see Theorem 1 below**) the integral on the left is also divergent. Now, If $s > 0$, then $\int_0^\infty e^{t(t-s)}dt \geq \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ ***does not possess a Laplace transform***.

The above example raises the question of what class or classes of functions possess a Laplace transform. Looking closely at ***Example 1(a)***, we notice that for $s > a$, the integral $\int_0^\infty e^{-(s-t)t}\,dt$ is convergent and a critical component for this convergence is the type of the function *f*(*t*): To be more specific, if *f*(*t*) is a continuous function such that

$|f(t)| \leq Me^{at}, \quad t \geq C \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Where $M \geq 0 \ \ and\ a\ and\ C\ are\ constants, then\ this\ condition\ yiels$

$$\int_0^\infty f(t)\,e^{-st}dt \leq \int_0^C f(t)\,e^{-st}dt + M\int_C^\infty e^{-(s-t)t}dt\,.$$

Since *f*(*t*) is continuous in $0 \leq t \leq C$; by letting $A = max\{|f(t)|: 0 \leq t \leq C\}$. We have

$$\int_0^C f(t)\,e^{-st}dt \leq A\int_0^C e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

On the other hand, Now, by ***Example 1(a)***, the integral $\int_C^\infty e^{-(s-t)t}dt$ is convergent for *s* > a. By the comparison theorem of improper integrals (**see Theorem 1 below**) the integral on the left is also convergent. That is, *f*(*t*) ***possesses a Laplace transform***.

Theorem 1(comparison theorem): Suppose that *f*(*t*) and *g*(*t*) are both integrable functions for all $t \geq t_0$ such that $|f(t)| \leq |g(t)|$ for $t \geq t_0$. If $\int_{t_0}^\infty g(t)dt$ is convergent, then $\int_{t_0}^\infty f(t)dt$ is also convergent. If, on the other hand, $\int_{t_0}^\infty f(t)dt$ is divergent then $\int_{t_0}^\infty g(t)dt$ is also divergent.

Definition: We call a function that satisfies the abovementioned condition in (**Eq.2**) a function with an ***exponential order at infinity***. *Graphically*, this means that the graph of *f*(*t*) is contained in the region bounded by the graphs of $y = Me^{at}$ and $y = -Me^{at}$, for $t \geq C$.

Note also that this type of functions controls the negative exponential in the transform integral so that to keep the integral from blowing up.
If $C = 0$ then we say that the function is ***exponentially bounded***.

Ex.2: Show that any bounded function $f(t)$ for $t \geq 0$ is exponentially bounded.

Solution: Since $f(t)$ is bounded for $t \geq 0$; there is a positive constant M such that $|f(t)| \leq M \; \forall \; t \geq 0$. But this is the same as in (Eq.2) with $a = 0$ and $C = 0$. Thus, $f(t)$ is exponentially bounded.

Another question that comes to mind is whether it is possible to relax the condition of continuity on the function $f(t)$. Let's look at the following situation.

Ex.3: Show that the square wave function whose graph is given in Figure 1 possesses a Laplace transform. Note that the function is periodic of period 2.

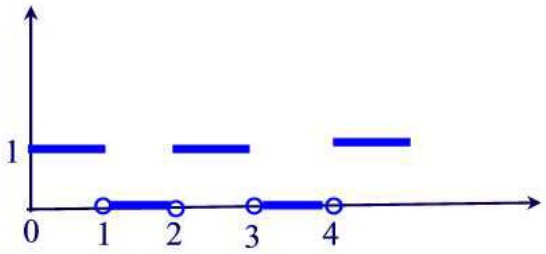


Fig.1

Solution: Since $f(t)e^{-st} \leq e^{-st}$, we have $\int_0^\infty f(t)e^{-st}dt \leq \int_0^\infty e^{-st}dt$, but the integral on the right convergent for $s > 0$ so that the integral on the left is convergent as well. That is, $L[f(t)]$ exists for $s > 0$.

The function of the above example belongs to a class of functions that we define next. ***You remember*** that we have defined a ***piecewise continuous*** function as follows:

Definition (*piecewise continuous*): A function is called ***piecewise continuous*** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (***jump discontinuities*** and ***no vertical asymptotes***) of each subinterval. Below is a sketch of a piecewise continuous function (**Fig 2**).

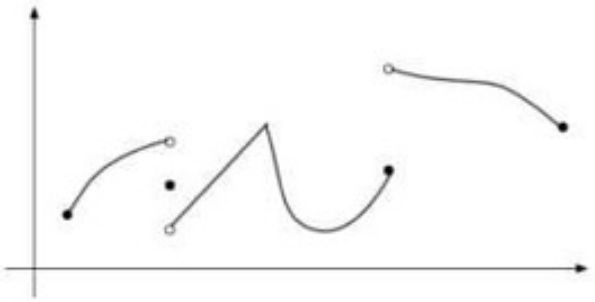


Fig 2.

Note that a piecewise continuous function is a function that has a finite number of breaks in it and does not blow up to infinity anywhere. A function defined for $\boldsymbol{t} \geq 0$ is said to be ***piecewise continuous on the infinite interval*** if it is piecewise continuous on $0 \leq \boldsymbol{t} \leq \boldsymbol{T}$ for all $\boldsymbol{T} > 0$.

Ex.4: Show that the following functions are piecewise continuous and of exponential order at infinity for
$\boldsymbol{t} \geq 0$.
(a) $f(t) = t^n$, $(b)\, f(t) = t^n \sin at$

Solution: **(*a*)** Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$; we have $t^n \leq n!\, e^t$. Hence $\boldsymbol{t^n}$ is piecewise continuous and exponentially bounded.

(*b*) Since $|t^n \sin at| \leq n!\, e^t$; we have $\boldsymbol{t^n \sin at}$ is piecewise continuous and exponentially bounded.

Next, we would like to establish the existence of the Laplace transform for all functions that are piecewise continuous and have exponential order at infinity.

Theorem 2* (*Existence*):** Suppose that *f*(*t*) is ***piecewise continuous on $\boldsymbol{t \geq 0}$ and has an ***exponential order at infinity with*** $\boldsymbol{|f(t)| \leq Me^{at}\ for\ t \geq C}$. Then the Laplace transform

$$F(s) = \int_0^{\infty} f(t)\, e^{-st} dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for *F*(*s*) to exist.

Proof: The integral in the definition of *F*(*s*) can be splitted into two integrals as

$$\int_0^{\infty} f(t)\, e^{-st} dt = \int_0^{C} f(t)\, e^{-st} dt + \int_C^{\infty} f(t)\, e^{-st} dt$$

Since *f*(*t*) is piecewise continuous in $0 \le t \le C$; it is bounded there.
By letting $A = max\{|f(t)|: 0 \le t \le C\}$; we have

$$\int_0^{C} f(t)\, e^{-st} dt \le A \int_0^{C} e^{-st} dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

Now, by ***Example 1***(a), the integral $\int_C^{\infty} f(t)\, e^{-st} dt$ is convergent for *s* > *a*. By ***Theorem 1*** the integral on the left is also convergent. That is, *f*(*t*) possesses a Laplace transform.

In what follows, we will denote the class of all piecewise continuous functions with exponential order at infinity by***PE*** . The next theorem shows that any linear combination of functions in ***PE*** is also in ***PE*** . The same is true for the product of two functions in ***PE*** .

Theorem 3*:** Suppose that *f*(*t*) and *g*(*t*) are two elements of ***PE with

$|f(t)| \le M_1 e^{a_1 t}$; $t \ge C_1$ *and* $|f(t)| \le M_2 e^{a_2 t}$; $t \ge C_2$.

(i) For any constants α and β the function $\boldsymbol{\alpha f(t) + \beta g(t)}$ is also a member of ***PE***. Moreover

$$L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$$

(ii) The function *h*(*t*) = *f*(*t*)*g*(*t*) is an element of ***PE***.

Proof:
(i) It is easy to see that $\boldsymbol{\alpha f(t) + \beta g(t)}$ is a piecewise continuous function.

Now, let $C = C_1 + C_2$; $a = max\{a_1, a_2\}, and\ M = |\alpha| M_1 + |\beta| M_2$ .Then for $t \ge C$ we have

$$|\alpha f(t) + \beta g(t)| \le |\alpha||f(t)| + |\beta||g(t)| \le |\alpha| M_1 e^{a_1 t} + |\beta| M_2 e^{a_2 t} \le M e^{at}.$$

This shows that $\boldsymbol{\alpha f(t) + \beta g(t)}$ is of exponential order at infinity. On the other hand,

$$L[\alpha f(t)+\beta g(t)] = \lim_{T\to\infty}\int_0^T [\alpha f(t)+\beta g(t)]dt = \alpha \lim_{T\to\infty}\int_0^T f(t)dt + \beta \lim_{T\to\infty}\int_0^T g(t)dt$$

$\alpha L[f(t)] + \beta L[g(t)]$.

(ii) It is clear that $h(t) = f(t)g(t)$ is a piecewise continuous function. Now,

Letting $C = C_1+C_2$; $M = M_1M_2$, and $a = a_1+a_2$ then we see that for $t \geq C$. we have

$$|h(t)| = |f(t)||g(t)| \leq M_1M_2e^{(a_1+a_2)t} = Me^{at}.$$

Hence, $h(t)$ is of exponential order at infinity. By ***Theorem 2*** , $L[h(t)]$ exists for $s > a$.

We next discuss the problem of ***how to determine the function*** $f(t)$ if $F(s)$ is given. That is, ***how do we invert the transform***? The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set ***PE*** and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in ***PE*** is unique.

Theorem 4: Let $f(t)$ and $g(t)$ be two elements in ***PE*** with Laplace transforms $F(s)$ and $G(s)$ such that
$F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

The standard techniques used to prove this theorem (i.e., complex analysis, residue computations, and/or Fourier's integral inversion theorem) are generally beyond the scope of an introductory differential equations course. The interested reader can find a proof in the book "***Operational Mathematics***" by Ruel Vance Churchill or in D.V. Widder "***The Laplace Transform***".

With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function f of exponential order at infinity whose Laplace transform is F; we call f the inverse Laplace transform of F and write $f = L^{-1}[F(s)]$. Symbolically

$$f(t) = L^{-1}[F(s)] \leftrightarrow F(s) = L[f(t)]$$

Ex.5: Find $L^{-1}\left[\frac{1}{s-1}\right]$, $s > 1$.

Solution: From ***Example 1***(a), we have that $L[e^{at}] = \frac{1}{s-a}$; $s > a$. In particular, for a =1 we find that $L[e^{t}] = \frac{1}{s-1}$; $s > 1$. Hence $L^{-1}\left[\frac{1}{s-1}\right] = e^{t}$; $t \geq 0$.

Remark: The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $L^{-1}[F(s)]$ we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.

When the function $f(t)$ is ***not continuous***, the uniqueness of the inverse Laplace transform is not assured. The following example addresses ***the uniqueness issue***.

Ex.6: Consider the two functions $f(t) = h(t)h(3-t)$ and $g(t) = h(t)h(t-3)$.

(a) Are the two functions identical?

(b) Show that $L[f(t)] = L[g(t)]$.

Solution:

(a) We have

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 3 \\ 0 & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1 & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

So the two functions are equal for all $t \neq 3$ and so they are ***not identical***.

(b) We have

$$L[f(t)] = L[g(t)] = \int_0^3 e^{-st}dt = \frac{1-e^{-3s}}{s} \; ; \; s > 0$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) ***where they are both continuous***.

The inverse Laplace transform possesses a linear property as indicated in the following result.

Theorem 5: Given two Laplace transforms $F(s)$ and $G(s)$ then for any constants a and b,

$$L^{-1}[aF(s) + bG(s)] = aL^{-1}[F(s)] + bL^{-1}[G(s)]$$

Proof: Suppose that $L[f(t)] = F(s)\ and\ \ L[g(t)] = G(s)$.

Since $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)] = aF(s) + bG(s);\ \ we\ have$

$$L^{-1}[aF(s) + bG(s)] = af(t) + bg(t) = aL^{-1}[F(s)] + bL^{-1}[G(s)].$$

Remark: Properties of the Laplace transform enable us to find Laplace transforms without having to compute them directly from the definition. We establish properties of Laplace transform that will be useful for solving ODEs.

Ex.7: (Laplace Transform of the Heaviside Step Function)

The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The following Figure displays the graph of h(t):

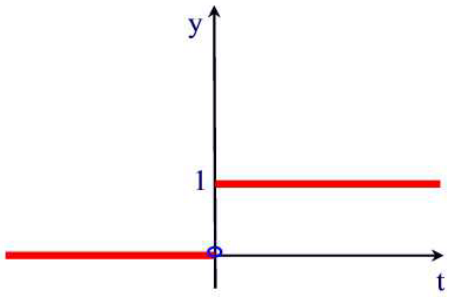


Fig: Heaviside Step Function

Taking the Laplace transform of $h(t)$ we find

$$L[h(t)] = \int_0^\infty h(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}\ ;s > 0$$

A Heaviside function at α > 0 is the shifted function $h(t-\alpha)$ (α units to the right). For this function, the Laplace transform is

$$L[h(t-\alpha)] = \int_0^\infty h(t-\alpha)e^{-st}\,dt = \int_\alpha^\infty e^{-st}\,dt = \left[-\frac{e^{-s\alpha}}{s}\right]_\alpha^\infty = \frac{e^{-s\alpha}}{s} \;\; ; s > 0$$

Ex.8: (Laplace Transform of *sin at* and *cos at*)

Using integration by parts twice we find

$$L[sin\ at] = \int_0^\infty \sin at\, e^{-st}\,dt = \int_0^\infty e^{-st}\,dt$$

$$= \left[-\frac{e^{-st}\sin at}{s} - \frac{ae^{-st}cos\ at}{s^2}\right]_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}sin\ at\ dt$$

$$= -\frac{a}{s^2} - \frac{a^2}{s^2}L[\sin at] \;\; ; \;\; then \;\; \left(\frac{s^2+a^2}{s^2}\right)L[\sin at] = \frac{a}{s^2} \;\; \rightarrow L[\sin at] = \frac{a}{s^2+a^2} \; ; s > 0 \ldots\ldots (\boldsymbol{A})$$

A similar argument shows that

$$L[\cos at] = \frac{s}{s^2+a^2} \; ; s > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\boldsymbol{B})$$

Ex.9: (Laplace Transform of *sinh at* and *cosh at*)

Using the linear property of *L* we can write

$$L[cosh\ at] = (L[e^{at}] + L[e^{-at}]) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \; ; s > |a| \ldots\ldots\ldots (C)$$

A similar argument shows that

$$L[cosh\ at] = \frac{a}{s^2-a^2} \; ; s > |a| \ldots\ldots\ldots (D)$$

Ex.10: (Laplace Transform of *polynomial*)

Let *n* be a positive integer. Using integration by parts we can write

$$\int_0^\infty t^n e^{-st}\,dt = -\left[\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt.$$

By repeated use of L'Hopital's rule we find $\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty} \frac{n!}{s^n e^{st}} = 0\,,\ for\ s > 0. Thus,$

$$L[t^n] = \frac{n}{s}L[t^{n-1}], \qquad s > 0.$$

Using induction on $n = 0,1,2,3,\ldots$ one can easily establish that

$$L[t^n] = \frac{n!}{s^{n+1}}, \qquad s > 0.$$

Using the above result together with the linearity property of *L* one can find the Laplace transform of any polynomial.

The next two results are referred to as the ***first*** and ***second shift theorems***. As with the linearity property, the shift theorems increase the number of functions for which we can easily find Laplace transforms.

Theorem 6: (*First Shifting Theorem*)

If *f*(*t*) is a piecewise continuous function for and $t \geq 0$ has exponential order at infinity with$|f(t)| \leq Me^{at}\,,\ t > C$; then for any real number α we have

$$L[e^{at}f(t)] = F(s-\alpha); s > a+\alpha, where\ F(s) = L[f(t)].$$

Proof: From the definition of the Laplace transform we have

$$L[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt.$$

Using the change of variable $\beta = s - a$, the previous equation reduces to

$$L[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-\beta t}f(t)dt = F(\beta)$$
$$= F(s-a), s > a+\alpha.$$

Theorem 7: (*Second Shifting Theorem*)
If *f*(*t*) is a piecewise continuous function for and $t \geq 0$ has exponential order at infinity with$|f(t)| \leq Me^{at}$, $t > C$; then for any real number α we have

$$L[f(t-\alpha)h(t-\alpha)] = e^{-\alpha s}F(s); s > a, where\ F(s) = L[f(t)]\ and\ h(t) is\ the\ Heaviside\ func.$$

Proof: From the definition of the Laplace transform we have

$$L[f(t-\alpha)h(s-\alpha)] = \int_0^\infty e^{-st}h(s-\alpha)f(t-\alpha)dt = \int_\alpha^\infty f(t-\alpha)e^{-st}dt.$$

Using the change of variable $\beta = t - \alpha$, the previous equation reduces to

$$L[f(t-\alpha)h(s-\alpha)] = \int_0^\infty f(\beta)e^{-s(\beta+\alpha)}d\beta = e^{-s\alpha}\int_0^\infty f(\beta)\,e^{-s\beta}d\beta = e^{-s\alpha}F(s); s > a.$$

Ex.11: Find (a) $L[e^{2t}t^2]$; $(b)L[e^{3t}\cos 2t]$; $(c)L^{-1}[e^{-2t}s^2]$.
Solution:
(a) By ***theorem 6*** , we have $L[e^{2t}t^2] = F(s-2); where\ \ L[t^2] = \frac{2!}{s^3} = F(s), s > 0$. Thus,

$$L[e^{2t}t^2] = \frac{2}{(s-2)^3}\ ; s > 2.$$

(b) As in part (a), we have $L[e^{3t}\cos 2t] = F(s-3); where\ L[\cos 2t] = F(s-3)$.
But $L[\cos 2t] = \frac{s}{s^2+4}\ ; s > 0. Thus, L[e^{3t}\cos 2t] = \frac{s-3}{(s-3)^2+4}\ ; s > 3.$

(c) Since $L[t] = \frac{1}{s^2}$, by ***theorem 7***, we have

$$\frac{e^{-2t}}{s^2} = L[(t-2)h(t-2)]. Therefore\ L^{-1}\left[\frac{e^{-2t}}{s^2}\right] = (t-2)h(t-2) = \begin{cases} 0 & 0 \leq t < 2 \\ t-2 & t \geq 2 \end{cases}.$$

The following result relates the Laplace transform of derivatives and integrals to the Laplace transform of the function itself.

Theorem 8: Suppose that *f*(*t*) is continuous for $t \geq 0$ and $f^{-1}(t)$ is a piecewise continuous of exponential order at infinity with $|f^{-1}(t)| \leq Me^{at}$, $t \geq C$; then

(1) *f*(*t*) is of exponential order at infinity.
(2) $L[f'(t)] = sL[f(t)] - f(0) = sF(s) - f(0);\ \ s > max\{a, 0\} + 1.$
(3) $L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0) = s^2F(s) - sf(0) - f(0);\ \ s > max\{a, 0\} + 1.$
(4) $L\left[\int_0^T f(u)\,du\right] = \frac{L[f(t)]}{s} = \frac{F(s)}{s}\ ;\ s > max\{a, 0\} + 1.$

Proof:

(1) By the fundamental theorem of calculus we have $f(t) = f(0) - \int_0^t f'(u)du.$
Also, since f' is piecewise continuous, $|f'(t)| \leq T\ ; for\ some\ T > 0\ and\ all\ 0 \leq t \leq c$. Thus,

$$|f(t)| = \left|f(0) - \int_0^t f'(u)du\right| = \left|f(0) - \int_0^C f'(u)du - \int_C^t f'(u)du\right|$$

$$\leq |f(0)| + TC + M\int_C^t e^{au}du$$

Note that if $a > 0, then$

$$\int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{e^{at}}{a}\ ;\ \ and\ so,$$

$$|f(t)| \leq \left[|f(0)| + TC + \frac{M}{a}\right]e^{at}.$$

$$If\ a = 0,; then\ \int_C^t e^{au}du = t - C.$$

$$Therefore, \qquad |f(t)| \leq |f(0)| + TC + M(t - C) \leq (|f(0)| + TC + M)e^t.$$

$$Now, if\ a < 0, then\ \int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{1}{|a|}.$$

$$So\ that\ |f(t)| \leq \left(|f(0)| + TC + \frac{M}{|a|}\right)e^t.$$

$$It\ follows\ that\ |f(t)| \leq Ne^{bt}\ ;\ \ t \geq 0, \qquad where\ b = max\{a, 0\} + 1$$

(2) From the definition of Laplace transform we can write

$$L[f'(t)] = \lim_{A\to\infty} \int_0^A f'(t)\, e^{-st} dt.$$

Since f' may have jump discontinuities at $t_1, t_2, \ldots t_N$ in the interval $0 \le t \le A$, we can write

$$\int_0^A f'(t)\, e^{-st} dt = \int_0^{t_1} f'(t)\, e^{-st} dt + \int_{t_1}^{t_2} f'(t)\, e^{-st} dt + \cdots \int_{t_N}^{A} f'(t)\, e^{-st} dt.$$

Integrating each term on the *RHS* by parts and using the continuity of *f*(*t*) to obtain

$$\int_0^{t_1} f'(t)\, e^{-st} dt = f(t_1)e^{-st_1} - f(0) + s\int_0^{t_1} f(t)\, e^{-st} dt.$$

$$\int_{t_1}^{t_2} f'(t)\, e^{-st} dt = f(t_2)e^{-st_2} - f(t_1)e^{-st_1} + s\int_{t_1}^{t_2} f(t)\, e^{-st} dt.$$

$$\int_{t_{N-1}}^{t_N} f'(t)\, e^{-st} dt = f(t_N)e^{-st_N} - f(t_{N-1})e^{-st_{N-1}} + s\int_{t_{N-1}}^{t_N} f(t)\, e^{-st} dt.$$

$$\int_{t_N}^{t_A} f'(t)\, e^{-st} dt = f(A)e^{-sA} - f(t_N)e^{-st_N} + s\int_{t_N}^{A} f(t)\, e^{-st} dt.$$

Also, by the continuity of f(t) we can write

$$\int_0^A f(t)\, e^{-st} dt = \int_0^{t_1} f(t)\, e^{-st} dt + \int_{t_1}^{t_2} f(t)\, e^{-st} dt + \cdots \int_{t_N}^{A} f(t)\, e^{-st} dt.$$

Hence,

$$\int_0^A f'(t)\, e^{-st} dt = f(A)e^{-sA} - f(0) + s\int_0^{A} f(t)\, e^{-st} dt.$$

Since *f*(*t*) has exponential order at infinity, $\lim_{A\to\infty} f(A)e^{-sA} = 0$. *Hence,*

$$L[f'(t)] = sL[f(t)] - f(0).$$

(3) Using part (2) we find

$$L[f''(t)] = sL[f'(t)] - f'(0) = s(sF(s) - f(0)) - f'(0).$$
$$= s^2F(s) - sf(0) - f'(0);\ \ s > max\{a, 0\} + 1.$$

(4) Since $\frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t);\ \ by\ part\ (2)\ we\ have,$

$$F(s) = L[f(t)] = s\left\{\int_0^t f(u)du\right\}\ ;\ \ and\ hence,$$

$$L\left\{\int_0^t f(u)du\right\} = \frac{L[f(t)]}{s} = \frac{F(s)}{s};\ \ s > max\{a, 0\} + 1.$$

Remark: The argument establishing part (2) of the previous theorem can be extended to higher order derivatives.

Theorem 8: Suppose that $f(t)\ , f'(t), f''(t), \ldots f^{(n-1)}(t)$ be continuous and $f^{(n)}(t)$ be piecewise continuous of exponential order at infinity with $\left|f^{(n)}(t)\right| \le Me^{at}\ ,\ t \ge C\ $; then

$$L\left[f^{(n)}(t)\right] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots f^{n-1}(0);\ \ s > max\{a, 0\} + 1.$$

We next illustrate the use of the previous theorem in solving initial value problems.

Ex.12: Solve the initial value problem

$$y'' - 4y' + 9y = t\ ;\ \ y(0) = 0\ ,\qquad y'(0) = 0.$$

Solution: We apply Theorem 8 that gives the Laplace transform of a derivative. By the linearity property of the Laplace transform we can write

$$L[y''] - 4L[y'] + 9L[y] = L[t].$$

Now since,

$L[y''] = s^2L[y] - sy(0) - y'(0) = s^2Y(s) - 1$

$L[y'] = sY(s) - y(0) = sY(s)$

$L[t] = \frac{1}{s^2}$

Where $L[y] = Y(s); we\ obtain$

$$s^2Y(s) - 1 - 4sY(s) + 9Y(s) = \frac{1}{s^2}$$

Rearranging gives

$$(s^2 - 4s + 9)Y(s) = \frac{s^2 + 1}{s^2}.\ Thus,$$

$$Y(s) = \frac{s^2 + 1}{s^2(s^2 - 4s + 9)}.$$

and

$$y(t) = L^{-1}\left[\frac{s^2 + 1}{s^2(s^2 - 4s + 9)}\right].$$

The Laplace Transform and the Method of Partial Fractions

In the ***Example12*** , we encountered the equation

$$y(t) = L^{-1}\left[\frac{s^2 + 1}{s^2(s^2 - 4s + 9)}\right].$$

We would like to find an explicit expression for *y*(*t*): This can be done using the method of partial fractions which is the topic of this section. According to this method, finding $L^{-1}\left(\frac{N(s)}{D(s)}\right)$, where *N*(*s*) and *D*(*s*) are polynomials, require decomposing the rational function into a sum of simpler expressions whose inverse Laplace transform can be recognized from a table of Laplace transform pairs.

The method of integration by partial fractions is a technique for integrating rational functions, i.e. functions of the form

$R(s) = \frac{N(s)}{D(s)}$; where $N(s)$ and $D(s)$ are polynomials.

The idea consists of writing the rational function as a sum of simpler fractions called partial fractions. This can be done in the following way:

Step 1. Use long division to find two polynomials *r*(*s*) and *q*(*s*) such that

$$\frac{N(s)}{D(s)} = q(s) + \frac{r(s)}{D(s)}.$$

Note that if the degree of $N(s)$ is smaller than that of $D(s)$ then $q(s) = 0$ and $r(s) = N(s)$.

Step 2. Write $D(s)$ as a product of factors of the form $(as + b)^n \; or \; (as^2 + bs + c)^n \; where \; as^2 + bs + c$ is irreducible , that is; $as^2 + bs + c = 0 \; has \; no \; real \; zeros$.

Step 3. Decompose $\frac{r(s)}{D(s)}$ into a sum of partial fractions in the following way:

(1) For each factor of the form $(s - \alpha)^k$ write

$$\frac{A_1}{(s-\alpha)} + \frac{A_2}{(s-\alpha)^2} + \cdots + \frac{A_k}{(s-\alpha)^k}; \text{ where the numbers } A1; A2; ___; Ak \text{ are to be determined.}$$

(2) For each factor of the form $(as^2 + bs + c)^k$, write

$$\frac{B_1 s + C_1}{as^2 + bs + c} + \frac{B_2 s + C_2}{(as^2 + bs + c)^2} + \cdots + \frac{B_k s + C_k}{(as^2 + bs + c)^k}$$

where the numbers $B1, B2, ___, Bk$ and $C1, C2, ___, Ck$ are to be determined.

Step 4. Multiply both sides by $D(s)$ and simplify. This leads to an expression of the form $r(s) = a \; polynomial$ whose coefficients are combinations of Ai; Bi; and Ci.

Finally, we find the constants, Ai;Bi; and Ci by equating the coefficients of like powers of s on both sides of the last equation.

Ex.13: Decompose into partial fractions $R(s) = \frac{s^3+s^2+2}{s^2-1}$.

Solution:

Step 1. $\frac{s^3+s^2+2}{s^2-1} = s + 1 + \frac{s+3}{s^2-1}$

Step 2. $s^2 - 1 = (s - 1)(s + 1)$

Step 3. $\frac{s+3}{(s-1)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)}$

Step 4. Multiply both sides of the last equation by $(s-1)(s+1)\ to\ obtain$

$s + 3 = A(s-1) + B(s+1)$

Expand the right hand side, collect terms with the same power of **s**, and identify coefficients of the polynomials obtained on both sides:

$s + 3 = (A+B)s + (B-A).$

$Hence\ A + B = 1\ and\ B - A = 3.\ Then\ A = -1\ and\ B = 2. Thus,$

$$\frac{s^3 + s^2 + 2}{s^2 - 1} = s + 1 - \frac{1}{s+1} + \frac{2}{s-1}.$$

Now, after decomposing the rational function into a sum of partial fractions all we need to do is to find the Laplace transform of expressions of the form

$$\frac{A}{(s-\alpha)^n} \quad or \quad \frac{Bs + C}{(as^2 + bs + c)^n}.$$

Ex.14: Fid $L^{-1}\left[\frac{1}{s(s-3)}\right]$.

Solution: We write

$$\frac{1}{s(s-3)} = \frac{A}{s} + \frac{B}{s-1}. Then\ 1 = A(s-3) + Bs\ or\ 1 = (A+B)s - 3A. Thus, A = -\frac{1}{3}; B = \frac{1}{3}.$$

Therefore,

$L^{-1}\left[\frac{1}{s(s-3)}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-3}\right] = -\frac{1}{3}h(t) + \frac{1}{3}e^{3t}\ ;\ t \geq 0\ and\ h(t)$is the Heaviside function.

Laplace Transforms of Periodic Functions

In many applications, the non-homogeneous term in a linear differential equation is a periodic function. In this section, we derive a formula for the Laplace transform of such periodic functions.

Recall that a function $f(t)$ is said to be T-periodic if we have $f(t+T) = f(t)$ whenever t and $t + T$ are in the domain of $f(t)$. For example, the sine and cosine functions are 2π-periodic whereas the tangent and cotangent functions are π-periodic.

If $f(t)$ is T-periodic for $t \geq 0$ then we define the function

$$f_T(t) = \begin{cases} f(t) & 0 \leq t < T \\ 0 & t \geq T \end{cases}.$$

The Laplace transform of this function is then

$$L[f_T(t)] = \int_0^{\infty} f_T(t)e^{-st}dt = \int_0^{T} f(t)\, e^{-st}dt.$$

The Laplace transform of a T−periodic function is given next.

Theorem 9: If $f(t)$ is a T−periodic, piecewise continuous function for $t \geq 0$ then

$$L[f(t)] = \frac{L[f_T(t)]}{1 - e^{-sT}} \;\; ; \;\; s > 0.$$

Proof: Since $f(t)$ is piecewise continuous, it is bounded on the interval $0 \leq t \leq T$. By periodicity, $f(t)$ is bounded for $t \geq 0$. Hence, it has an exponential order at infinity. By ***Theorem 2***, $L[f(t)]$ exists for $s > 0$: Thus,

$$L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = \sum_{n=0}^{\infty} \int_0^{T} f_T(t - nT)h(t - nT)e^{-st}dt,$$

where the last sum is the result of decomposing the improper integral into a sum of integrals over the constituent periods. By the Second Shifting Theorem (i.e. ***Theorem 7***) we have

$$L[f_T(t - nT)h(t - nT)] = e^{-nTs}L[f_T(t)] \;\; ; s > 0.$$

Hence,

$$L[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}L[f_T(t)] = L[f_T(t)]\left(\sum_{n=0}^{\infty} e^{-nTs}\right).$$

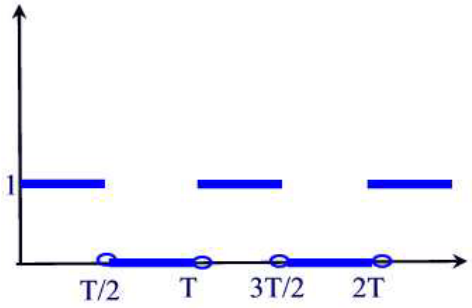
Since $s > 0$; it follows that $0 < e^{-nTs} < 1$ so the series $\sum_{n=0}^{\infty} e^{-nTs}$ is a convergent geoemetric series with limit $\frac{1}{1-e^{-sT}}$. $Therefore,$

$$L[f(t)] = \frac{L[f_T(t)]}{1 - e^{-sT}} \;\; ; \;\; s > 0.$$

Ex.15: Determine the Laplace transform of the function

$$f(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases}$$

Solution: The graph of *f*(*t*) is shown in the following Figure



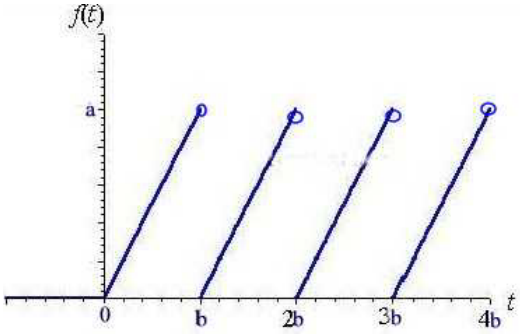
By ***Theorem 9***,

$$L[f(t)] = \frac{\int_0^{\frac{T}{2}} e^{-st}\, dt}{1 - e^{-sT}} \;\; ; \;\; s > 0.$$

Evaluating this last integral, we find

$$L[f(t)] = \frac{\frac{1 - e^{-\frac{sT}{2}}}{s}}{1 - e^{-sT}} = \frac{1}{s\left(1 + e^{-\frac{sT}{2}}\right)} \;\; ; \;\; s > 0.$$

Ex.16: Find the Laplace transform of the sawtooth curve shown in the following Figure



Solution: The given function is periodic of period b. For the first period the function is defined by

$$f_b(t) = \frac{a}{b}t[h(t) - h(t-b)].$$

$$L[f_b(t)] = L\left[\frac{a}{b}t(h(t) - h(t-b))\right] = -\frac{a}{b}\frac{d}{ds}L[h(t) - h(t-b)].\ But,$$

$$L[h(t) - h(t-b)] = L[h(t)] - L[h(t-b)] = \frac{1}{s} - \frac{e^{-bs}}{s}\ ;\ \ s > 0.$$

$Hence,$

$$L[f_b(t)] = \frac{a}{b}\left(\frac{1}{s^2} - \frac{bse^{-bs} + e^{-bs}}{s^2}\right).$$

$Finally,$

$$L[f(t)] = \frac{L[f_b(t)]}{1 - e^{-bs}} = \frac{a}{b}\left[\frac{1 - e^{-bs} - bse^{-bs}}{s^2(1 - e^{-bs})}\right].$$

Ex.17: Find $L^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}\right]$.

Solution: Note first that

$$\frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})} = \frac{1 - e^{-s} - se^{-s}}{s^2(1 - e^{-s})}.$$

According to the previous example with $a = 1$ and $b = 1$ we find that

$L^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}\right]$ is the sawtooth function shown in the above Figure .

Laplace transform of partial derivative

We know that Laplace transform of the function $g(t)$ is

$$L[g(t)] = G(s) = \int_0^t g(t)\, e^{-st} dt, \qquad Re\ s > 0 \tag{1}$$

and the inverse Laplace transform is given by

$$L^{-1}[G(s)] = g(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st}\, g(s) ds, \ \ \alpha > 0 \tag{2}$$

where L and L^{-1} are linear operators.

To obtain Laplace transform of partial derivative we use integration by parts, and then we have:

$$L\left[\frac{\partial g(x,t)}{\partial t}\right] = sG(x,s) - g(x,0), \tag{3}$$

$$L\left[\frac{\partial^2 g(x,t)}{\partial t^2}\right] = s^2 G(x,s) - sg(x,0) - \frac{\partial g(x,0)}{\partial t}, \tag{4}$$

$$L\left[\frac{\partial^3 g(x,t)}{\partial t^3}\right] = s^3 G(x,s) - s^2 g(x,0) - s\frac{\partial g(x,0)}{\partial t} - \frac{\partial^2 g(x,0)}{\partial t^2}, \tag{5}$$

$$L\left[\frac{\partial^n g(x,t)}{\partial t^n}\right] = s^n G(x,s) - s^{n-1} g(x,0) - s^{n-2}\frac{\partial g(x,0)}{\partial t} - \cdots s^2 \frac{\partial^{n-3} g(x,0)}{\partial t^{n-3}} - s\frac{\partial^{n-2} g(x,0)}{\partial t^{n-2}} - \frac{\partial^{n-1} g(x,0)}{\partial t^{n-1}} \tag{6}$$

$$L\left[\frac{\partial g(x,t)}{\partial x}\right] = \frac{d}{dx}[G(x,s)], \tag{7}$$

$$L\left[\frac{\partial^2 g(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}[G(x,s)], \tag{8}$$

$$L\left[\frac{\partial^3 g(x,t)}{\partial x^3}\right] = \frac{d^3}{dx^3}[G(x,s)], \tag{9}$$

where $G(x,s)$ is the Laplace transform of $g(x,t)$.

We can easily extend this result to the nth partial derivative by using mathematical induction.

Wavelet Transform and Multiresolution Analysis

In this chapter we introduce the basic idea of the wavelet transform (WT) and its properties, as well as, its applications. Then the concept of Multiresolution Analysis (MRA) is explained.

Wavelet Transform (WT)

For the analysis of a non-stationary (transient) signal, using the FT does not give suitable results since sinusoids are smooth and predictable and are good at describing constant frequencies (stationary) signals.
Wavelets are better at describing irregularities, and other events that start and stop within the signal. Therefore, wavelet analysis (or the wavelet transform) is a successful approach to the problem of analyzing a non-stationary signal both in time and in frequency.

Wavelets are mathematical functions defined over a finite interval and having an average value of zero that transform data into different frequency components, representing each component with a resolution matched to its scale. It was first introduced by A. Grossmann and J. Morlet in 1984. The basic idea of the (WT) is to represent any arbitrary function $f(t)$ as a superposition of a set of such wavelets or basis functions. These basis functions are obtained from a single wavelet called ***mother wavelet*** $\psi(t)$ because all other wavelet functions within the family are obtained by ***dilating*** and ***translating*** of $\psi(t)$ by amounts s and τ respectively as given bellow:

$$\psi_{s,\tau}(t) = \left\{\psi\left(\frac{t-\tau}{s}\right) ;\ (s,\tau) \in \mathbb{R}^+ \times \mathbb{R}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

The process of changing the two parameters s and τ that result in the basis functions are shown in **Figure 1**.

These wavelets are distinguishing by compactly supported functions defined over a finite interval and having an average value of zero, and that leads to efficient implementation. The mother wavelet $\psi(t)$ is the function with zero translation and a dilation of one.

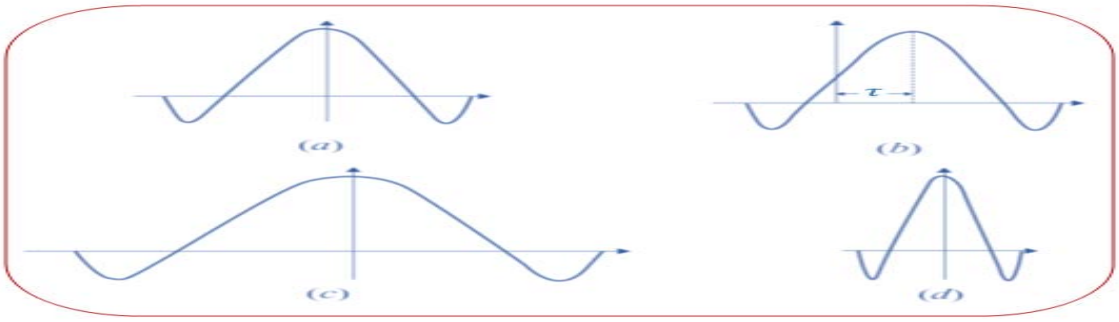


Figure 1: Effect of Time Dilation and Translation on the Mother Wavelet

(a) Mother wavelet $\psi(t) = \psi_{1,0}$; $s = 1, \tau = 0$. **(b)** Wavelet $\psi_{1,\tau}(t)$; $s = 1, \tau \neq 0$.

(c) Wavelet $\psi_{2,0}(t)$; $at\ scale\ s = 2, \tau = 0$. **(d)** Wavelet $\psi_{0.5,0}(t)$; $at\ scale\ s = 0.5, \tau = 0$.

In wavelet transform the basis functions are **wavelets**. Wavelets tend to be irregular and symmetric. All wavelet functions, $\psi(2st - \tau)$, are derived from a single mother wavelet, $\psi(t)$. This wavelet is a small wave or pulse like the one shown in **Figure 2**.

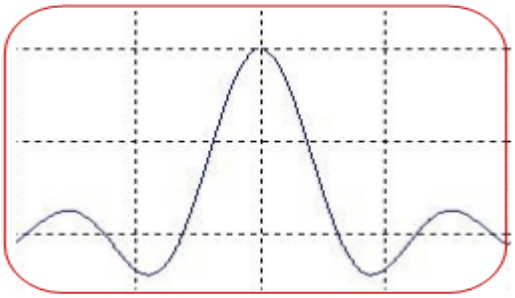


Figure 2: Mother Wavelet $\psi(t)$.

Normally it starts at time t = 0 and ends at $t = T$. The ***shifted*** wavelet $\psi(t - \tau)$ starts at $t = \tau$ and ends at $t = \tau + T$. The ***scaled*** wavelets $\psi(2st)$ start at t = 0 and end at $t = T/2s$. Their graphs are $\psi(t)$ compressed by the factor of 2s as shown in **Figure 3**. For example, when s = 1, the wavelet is shown in **Figure 3(a)**. If s = 2 and 3, they are shown in (b) and (c), respectively.

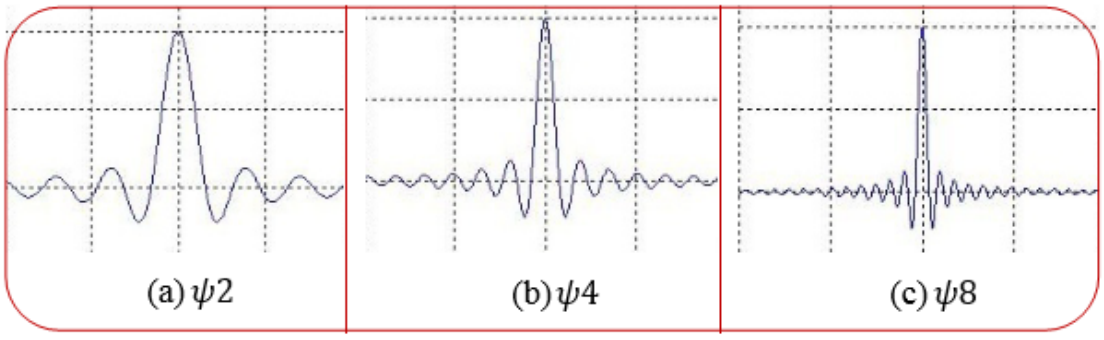


Figure 3: Scaling the Wavelets

By a dilation we mean a scaling of the argument; so, given any function $\psi(t)$ and a parameter $s>0$, $\psi(\frac{t-\tau}{s})$ is a dilation of $\psi(t)$. Consequently, a dilation of a function corresponds to either a spreading out or contraction of the function. The factor $\frac{\mathbf{1}}{\sqrt{|s|}}$ is introduced with $\psi_{s,\tau}(t)$ equation to keep the energy of the mother wavelets constant that it yields normalization necessary to have an orthonormal wavelet basis.

The translation simply means a shift of the argument along the real axis, that is, given, the translation of $\psi(t)$ by τ is $\psi(t-\tau)$. For any analyzing wavelet $\psi(t)$ we thus define a family of functions $\psi_{s,\tau}(t)$ by the dilations and translations of $\psi(t)$ as given by the Eq. (2):

$$\psi_{s,\tau}(t) = \left\{\frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right) ;\ \ s,\tau \in \mathbb{R}^{+} \times \mathbb{R}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Each $\psi_{s,\tau}$ is called a wavelet, where:

$\psi(t)$ is a mother wavelet,
s : coefficient of expansions or scaling (dilation).
τ : coefficient of translation.

The translation τ and dilation s allow the wavelet transform to be localized in time and frequency. Also, wavelet basis functions can represent functions with discontinuities and spikes in a more compact way than sine and cosine.

The wavelets are called ***orthogonal*** when their inner products are zero. The smaller the scaling factor is, the wider the wavelet is. Wide wavelets are comparable to low-frequency sinusoids and narrow wavelets are comparable to high frequency sinusoids.

The following are some important definitions which are related to the wavelet transform.

► Translation

Let $f: \mathbb{R} \to \mathbb{R}$ and $\in \mathbb{R}$, then the translation $T_y: \mathbb{R} \to \mathbb{R}$, is an operator defined by
$T_y f(x) = f(x-y)$

► t-Dilation

Let $f: \mathbb{R} \to \mathbb{R}$ and $t > 0$, then the t-dilation $f_t: \mathbb{R} \to \mathbb{R}$ is defined by

$$f_t(x) = \frac{1}{t} f\left(\frac{x}{t}\right)$$

► **Orthogonally**

The expansion functions $\{\psi_{s,\tau}(x)\}$ form an orthogonal of basis functions:

$$\langle \psi_{s,\tau}(x), \psi_{s_1,\tau_1}(x) \rangle = \int \psi_{s,\tau}(x)\, \psi_{s_1,\tau_1}(x) dx$$

$$= \begin{cases} 1 & for \quad s = s_1, \ \tau = \tau_1 \\ 0 & for \quad s \neq s_1, \ \tau \neq \tau_1 \end{cases}$$

where

$\psi_{s,\tau}(x)$: is a decomposition wavelet,
s and s_1: are dilation (scaling) factors,
τ and τ_1 : are translating (shifting) factors.

► **Compact Support**

We say that $\psi(x)$ has compact support on interval *I*, if it has zero values (vanish) outside this interval, so it is limited in time domain.

► **Admissibility Condition**

The inverse wavelet transform is held if the wavelet function is satisfied for any $\psi(x)$, then ;

where $\Psi(x)$ is the Fourier transform of the fundamental mother wavelet $\psi(x)$, $\Psi(\omega) = \int_{-\infty}^{\infty} \psi(x)\, e^{-j\omega x} dx$, and *c* represents how closely correlated of the wavelet with this section of the signal, so the higher value of *c* is the more of similarity. Then *c* is positive and finite, in most cases, this simply means that $\Psi(0) = 0$ and $\Psi(\omega) \to 0 \; as \; \omega \to \infty$ fast enough to make $c < \infty$.
The requirement that *c* be positive and finite imposes another restriction of the choice of wavelet.

► Normalized

A wavelet function is defined as a function with a zero average

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \;\; and \;\; \|\psi(x)\| = 1$$

It is normalized, and centered in the neighborhood for $x = 0$.

► Series Expansion

A function or signal $f(x)$ can often be better analyzed as a linear combination of expansion function

$$f(x) = \sum_k \alpha_k \emptyset_k(x) \dots\dots\dots\dots\dots\dots\dots (3)$$

where k is an integer index of the finite or infinite sum,
α_k are real valued expansion coefficients,
$\emptyset_k(x)$ are real valued expansion functions and called (basis functions).
The expressible functions form a function space that is referred to as the closed span of the expansion set, denoted by:

$$V = \overline{span\{\emptyset_k(x)\}} \dots\dots\dots\dots\dots (4)$$

► Scaling Functions

The set of expansion functions composed of integer translations and binary scaling of the real, square integrable function $\emptyset(x)$; is the set $\{\emptyset_{s,\tau}(x)\}$, where

$$\emptyset_{s,\tau}(x) = 2^{s/2}\emptyset(2^s x - \tau), \forall s, \tau \in \mathbb{Z} \; and \; \emptyset(x) \in L^2(\mathbb{R}) \dots\dots\dots\dots\dots (5)$$

where, s : is $\emptyset_{s,\tau}(x)$'s width, how broad or narrow it is along x-axis,

τ : is the position of $\emptyset_{s,\tau}(x)$ along the x-axis,

$2^{\frac{s}{2}}$: is controls $\emptyset_{s,\tau}(x)$'s height or amplitude.

Because the shape of $\emptyset_{s,\tau}(x)$ changes with s, $\emptyset(x)$ is called a ***scaling function***.

By the Eq. (4), we will denote the subspaces spanned over τ for any s as

$$V_s = \overline{span_\tau\{\emptyset_{s,\tau}(x)\}} \dots\dots\dots\dots\dots\dots\dots (6)$$

A V_s sequence of nested closed subspaces functions spanned by $\emptyset_{s,\tau}$ over τ for any s; by Eq. (3), if $f(x) \in V_s$ then, it can be written as

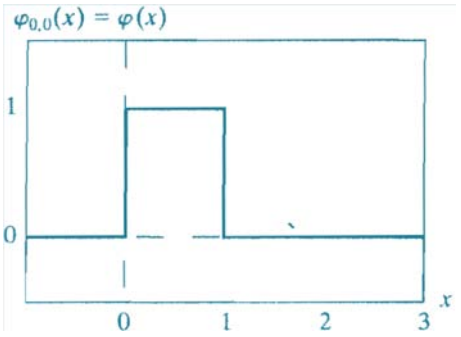
$$f(x) = \sum_{\tau} \alpha_{\tau} \emptyset_{s,\tau}(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

It is noted that increasing s increases the size of V_s , which implies that functions with smaller variations will be included in the subspace. This is because when s is increasing, the set $\{\emptyset_{s,\tau}(x)\}$ becomes narrower and separated by smaller changes in x, as will be seen in the following example:

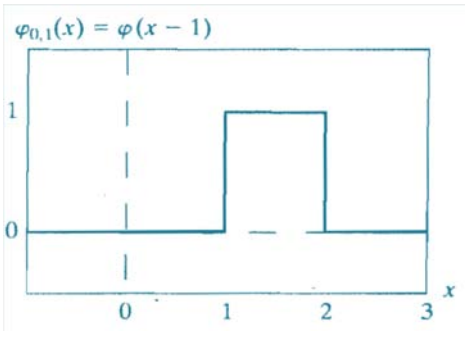
Example 1: Consider the unit-height, unit-width scaling function $\emptyset(x)$, which is called **Haar scaling function**, where;

$$\emptyset(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

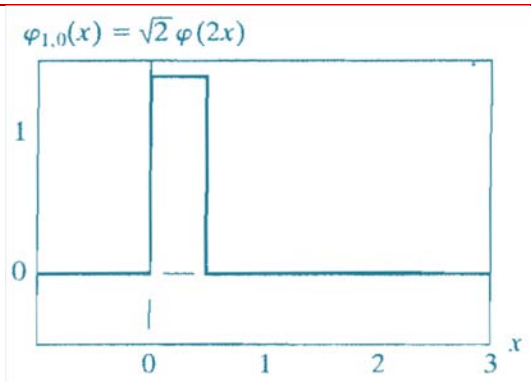
Figures (4, a-d) depicts some of the many expansion functions by substituting the given scaling function into Eq. (5).



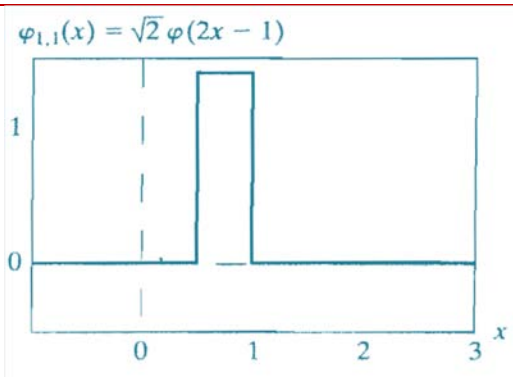
(a)



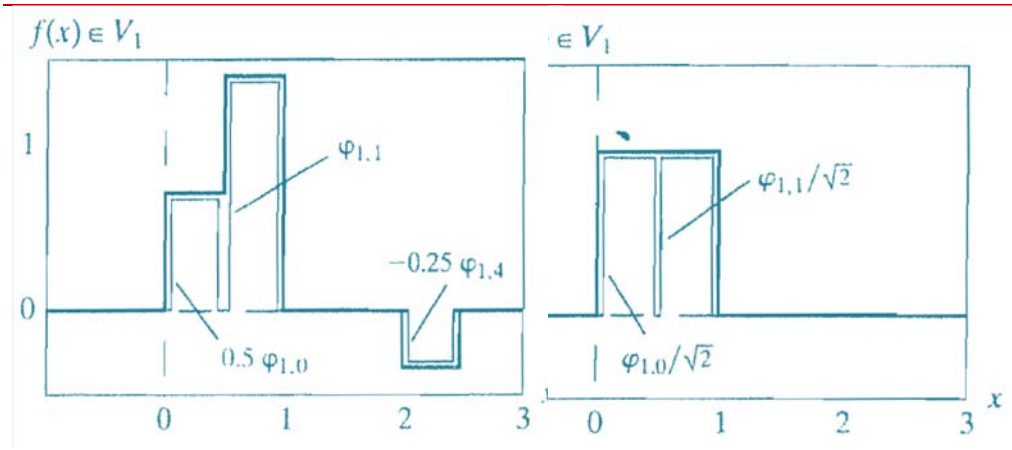
(b)



(c)



(d)



(e) (f)

Figures 4 : Some Haar Scaling Functions.

Note that the expansion functions for s = 1 in Figures 4(c) and (d) are half as wide as those for s = 0 in Figures 4(a) and (b). In addition, it can be defined twice as many V_1 scaling functions as V_0 scaling functions (for example: $\emptyset_{1,0}$ and $\emptyset_{1,1}$ of V_1 versus $\emptyset_{0,0}$ of V_0 for $\in [0,1)$).

Figure 4(e) shows a member of subspace V_1 . This function does not belong to V_0 because the V_0 expansion functions in Figures 4(a) and (b) are too coarse to represent it. Higher resolution functions like those in Figures 4(c) and (d) are required. They can be used, as shown in (e), to represent the function by the expansion

$$f(x) = \frac{1}{2}\emptyset_{1,0}(x) + \emptyset_{1,1}(x) - \frac{1}{4}\emptyset_{1,4}(x) \qquad (9)$$

Also, the decomposition of $\emptyset_{0,0}(x)$ as a sum of V_1 expansion functions is included in Figure 4(f). Similarly, any V_0 expansion function can be decomposed using the following relation:

$$\emptyset_{0,\tau}(x) = \frac{1}{\sqrt{2}}\emptyset_{1,2\tau}(x) + \frac{1}{\sqrt{2}}\emptyset_{1,2\tau+1}(x) \qquad (10)$$

Thus, if $f(x) \in V_0$ then $f(x) \in V_1$. This means $V_0 \subset V_1$.

► **Wavelet Function**

A wavelet function $\psi_{s,\tau}(x)$ can be defined as spans for the difference between any two adjacent scaling subspaces, V_s and V_{s+1}.

The set $\psi_{s,\tau}(x)$ of wavelets can be defined as a basis wavelet from the mother function $\psi(x)$ such as:

$$\psi_{s,\tau}(x) = 2^{\frac{s}{2}}\psi(2^s x - \tau) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

where $s, \tau \in \mathbb{Z}$ and $\psi(x) \in L^2(\mathbb{R})$ that spans the W_s spaces as:

$$W_s = \overline{span_\tau\{\psi_{s,\tau}(x)\}} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Where W_s is a sequence of closed subspaces of functions spanned by $\psi_{s,\tau}(x)$ over τ for any s, by Eq. 3, if $f(x) \in W_s$ then, it can be written as

$$f(x) = \sum_\tau \alpha_\tau \psi_{s,\tau}(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Remark 1: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales. So, the subspaces containing high-resolution function must also contain all lower resolution functions, that is,

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty = L^2(R) \ \ldots\ldots\ldots\ldots\ldots (14)$$

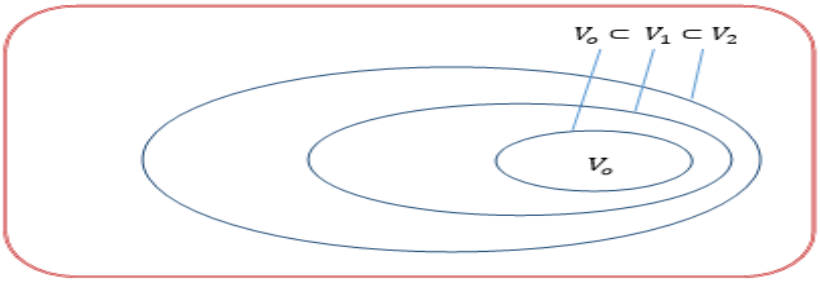


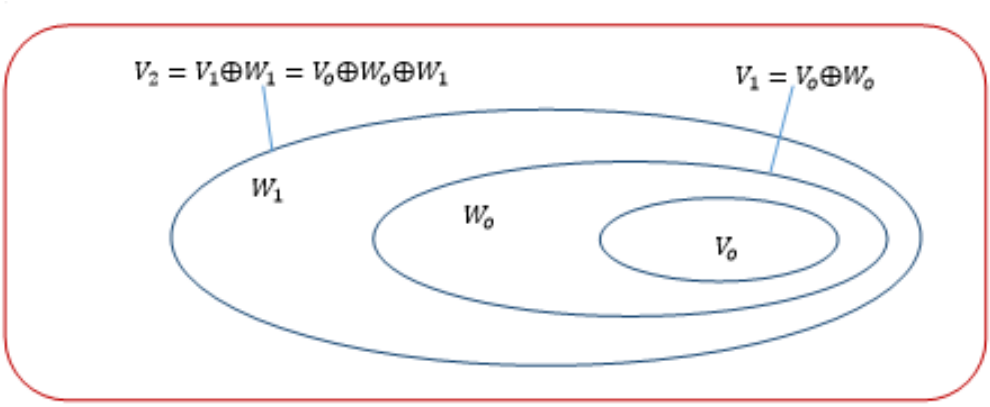
Figure 5: The Nested Function Spaces Spanned By a Scaling Function.

Remark 2: The scaling and wavelet functions subspace are related by

$$V_{s+1} = V_s \oplus W_s \ \ and \ \ L^2(R) = V_{s_0} \oplus W_{s_0} \oplus W_{s_0+1} \oplus \cdots \qquad (15)$$

Where $\oplus$ denotes the union of spaces (like the union of sets) and $V_s \perp W_s$.

The Relationship Between Scaling and Wavelet Function is explained in the following Figure :



The Relationship Between Scaling and Wavelet Function Space

Multiresolution Analysis (MRA) **:**

As it name implies, the multiresolution analysis is interested with the representation and analysis of signals at more than one resolution. It is to approximate a function *f(x)* at various levels of resolution by analyze a function at different scales. In **multiresolution analysis**, two functions are taking into consideration: the scaling function $\phi(x)$, which structures, using Eq. (5), a number of scaling functions $\phi_{s,\tau}(x)$ by the dilated (scaled) and translated (shifted), and the mother wavelet $\psi(x)$, which structures, using Eq. (11), a number of orthogonal wavelet basis functions $\psi_{s,\tau}(x)$.

It is clear that $\|\psi_{s,\tau}\|_{L^2}^2 = \|\psi\|_{L^2}^2$, because if we let

$u = (2^s x - \tau) \Longrightarrow du = 2^s dx \Longrightarrow dx = \frac{du}{2^s} \Longrightarrow dx = 2^{-s} du$, we obtain

$$\|\psi_{s,\tau}\|_{L^2}^2 = \int_{\mathbb{R}} 2^s |\psi(u)|^2 \frac{du}{2^s} = \int_{\mathbb{R}} |\psi(u)|^2 \, du = \|\psi\|_{L^2}^2.$$

Similarly, $\|\phi_{s,\tau}\|_{L^2}^2 = \|\phi\|_{L^2}^2$. In fact, $\|\phi_{s,\tau}\|_{L^2}^2 = \|\psi_{s,\tau}\|_{L^2}^2$, $\forall s, \tau \in \mathbb{Z}$.

To design a multiresolution analysis, we need a set of nested spaces, by selecting the functions $\phi_{s,\tau}, \forall s, \tau \in \mathbb{Z}$; we determine the nested spaces V_s , $\forall s \in \mathbb{Z}$. For fixed s , the set of scaling functions $\phi_{s,\tau}$ are orthonormal.

Now consider V_s to be the vector spaces corresponding to spanning set $\{\phi_{s,\tau}(x)\}$, assuming that the resolution increases with decreasing s, and these vector spaces characterize successive approximation vector spaces, (i.e, each space V_s is contained in the

next resolution space V_{s+1}), this is depicted in **Figure 5** and each with resolution 2^s such as in Eq.(14). To say that $f(x) \in V$ means that *f*(x) belongs to the closed span of $\phi_{s,\tau}(x)$ and can be written in the form of Eq. (7).
Now, a (*MRA*) with scaling function ϕ consists of a sequence of closed subspaces $\{V_s\ ,\ \ s \in \mathbb{Z}\}$ of $L^2(R)$ which have the following properties:

1. $V_s \subset V_{s+1}$, $\forall s \in \mathbb{Z}$. (Nested)
2. $\overline{\bigcup_{s\in\mathbb{Z}} V_s} = L^2(\mathbb{R})$. (Density)
3. $\bigcap_{s\in\mathbb{Z}} V_s = \{0\}$. (Trivial intersection)
4. The following scale relations exist:
$f(x) \in V_s \leftrightarrow f(2x) \in V_{s+1}$. (Scaling invariance property)

$f(x) \in V_0 \leftrightarrow f(x-n) \in V_0$, $\forall n \in \mathbb{Z}$. (Scaling invariance property)

5. There exists a scaling function $\phi \in V_0$ such that its integer translates, $\{\phi(x-n)\}$ is an orthonormal basis of V_0, where $\phi_{s,\tau}(x)$ as in Eq. (5).

This means that the basic rule of multiresolution analysis is that whenever the above properties are satisfied, there exists an orthonormal wavelet basis and scaling basis such that any $f(x) \in L^2(\mathbb{R})$ can be expanded as a linear combination of both the scaling basis function $\phi_{s,\tau}(x)$ and the wavelet basis functions $\psi_{s,\tau}(x)$.

In a (MRA), since $\phi(x) \in V_0 \subset V_1$ and $\{\phi_{s,\tau}(x), \tau \in \mathbb{Z}\}$ is an orthonormal basis of V_1, there exists some set of coefficients $\{a_n\ , n \in \mathbb{Z}\}$ such that the function $\phi(x) \in V_0$ can be represented as a linear combination of the functions from V_1 and so,

$$\phi(x) = \sum_n a_n\, \phi(2x-n) \ldots\ldots\ldots\ldots (16)$$

Alternatively, if we consider property (4: Scaling invariance property) of (MRA) written as a linear combination of $\phi(x)$ in the scaled form defined in Eq. (16), the recursion for $\phi(x)$ can be written in terms of a new set of coefficients $\{h_n\}$ as:

$$\phi(x) = \sqrt{2}\sum_n h_n\, \phi(2x-n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

where

$$\sum_n a_n = \sqrt{2}\sum_n h_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

for some coefficients $\{h_n, n \in \mathbb{Z}\}$, using the fact that $\{\phi_{s,\tau}(x)\}$ are orthonormal. The coefficients $\{h_n, n \in \mathbb{Z}\}$ can be obtained by computing the inner product:

$$h_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(x)\, \phi(2x - n)\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

where the function $\phi(x)$ is the scaling function.

It is interesting to notice that the scaling function $\phi(x)$ can be recursively generated by scaled (shrunk to half) and shifted versions of itself, as it is described by Eq. (17). This means that the scaling function $\phi(x)$ has the self similarity property.
For each (*MRA*), it is also possible to define a mother wavelet, $\psi(x)$, which will explain the detail at each level *s* . Assume W_s to be the orthogonal complement of V_s in V_{s+1} ; that is W_s is the difference between the function space V_{s+1} spanned by scaling functions $\phi_{s+1,\tau}(x)$; and the function space V_s spanned by $\phi_{s,\tau}(x)$, so that

$$V_{s+1} = V_s \oplus W_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

where $\oplus$ represents the union of the two spaces. The space W_s is composed of all functions representable in V_{s+1} but not representable in V_s . This can be carried out recursively to get:

$$L^2(\mathbb{R}) = V_\infty = V_s \oplus W_s \oplus W_{s+1} \oplus W_{s+2} \oplus \ldots \qquad (21)$$

Similar to a function space V_s spanned by the scaling functions $\phi_{s,\tau}(x)$, the function space W_s is also spanned by a set of basis function, called the wavelet functions. Then the fundamental result is that $\{\psi(x - \tau),\ \tau \in \mathbb{Z}\}$ forms an orthonormal basis for W_s , and $W_0 \subset V_1$, orthogonal to all functions in V_s , where $\psi_{s,\tau}(x)$ is defined as in Eq. (11).

The wavelet functions $\psi_{s,\tau}(x)$ can be expanded in the space V_{s+1} as:

$$\psi_{s,\tau}(x) = \sum_n g_n\, \phi_{s+1,n}(x) = \sum_n g_n\, 2^{(s+1)/2} \phi(2^{s+1}x - n) \ldots\ldots (22)$$

where g_n are the expansion coefficients. Usually we let $s = 0$ and drop the subscripts *s* and τ to indicate that any wavelet function $\psi_s(x) \in V_1$ can be expressed as a linear combination of the basis scaling functions ϕ_{s+1} of the functions from V_1 ;

$$\psi(x) = \sqrt{2} \sum_n g_n\, \phi(2x - n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

This is in the same form for the scaling functions Eq.(17).

The Eq. (23) can be expressed as:

$$\psi(x) = \sum_{n} (-1)^n \, a_{1-k} \phi(2x - n) \; \ldots\ldots\ldots (24)$$

where

$$g_n = (-1)^n 2^{-1/2} a_{1-k} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The coefficients $\{g_n\}$ can be obtained by computing the inner product:

$$g_n = \sqrt{2} \int_{-\infty}^{\infty} \psi(x) \, \phi(2x - \tau) dx \; \ldots\ldots\ldots\ldots\ldots (26)$$

These coefficients are called coefficients of highpass filter. Coefficients of highpass filter can be calculated from coefficients of lowpass filter using this Equation:

$$g_n = (-1)^n h_{1-n} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

Remark 3: The scaling function in example (1) satisfies the requirements of (*MRA*).

Example 2: The Haar scaling coefficients was defined as:

$h_\phi(0) = h_\phi(1) = \frac{1}{\sqrt{2}}$. Using Eq.(27), the coefficients for the wavelet function are:

$g_0 = (-1)^0 h_{1-0} = h_1 = \frac{1}{\sqrt{2}}$ and $g_1 = (-1)^1 h_{1-1} = -h_0 = -\frac{1}{\sqrt{2}}$.

Substituting these values into Eq.(23), we obtained the wavelet functions as:

$$\psi(x) = \sqrt{2} \sum_{n} g_n \, \phi(2x - n) = \sqrt{2} \frac{1}{\sqrt{2}} \phi(2x) - \sqrt{2} \frac{1}{\sqrt{2}} \phi(2x - 1)$$

$$= \phi(2x) - \phi(2x - 1).$$

which is plotted in Figure 6(a), thus, the Haar wavelet function is

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & otherwise \end{cases}$$

Using Eq. (11), we can generate the universe of scaled and translated Haar. Figure 6(a) explains $\psi_{0,0}(x) = 2^0\psi(2^0x - 0) = \psi(x)$, and two such wavelets $\psi_{0,2}(x)$ and $\psi_{1,0}(x)$ are plotted in Figure 6(b) and Figure 6(c) respectively, where;

$$\psi_{0,2}(x) = 2^0\psi(2^0x - 2) = \psi(x - 2)$$

$$\psi_{1,0}(x) = 2^{\frac{1}{2}}\psi(2^1x) = \sqrt{2}\psi(2x)$$

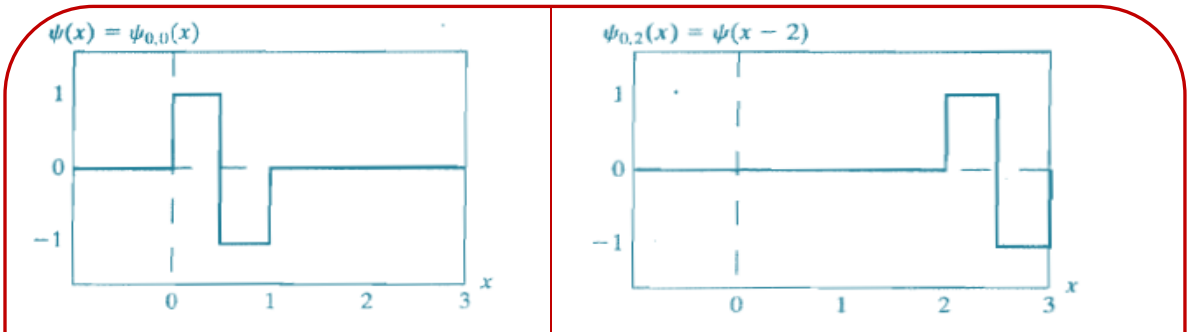
Finally, Figure 6(d) shows a function of space V_1 that is not in the subspace V_0. However, Eq.(20) indicates that it can be expanded using V_0 and W_0 expansion functions as:

$$f(x) = f_a(x) + f_d(x) \ \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

where

$$f_a(x) = \frac{3\sqrt{2}}{4}\phi_{0,0}(x) - \frac{\sqrt{2}}{8}\phi_{0,2}(x) \text{ and } f_d(x) = \frac{-\sqrt{2}}{4}\psi_{0,0}(x) - \frac{\sqrt{2}}{8}\psi_{0,2}(x).$$

Here, $f_a(x)$ is an approximation of $f(x)$ using V_0 scaling functions, while $f_d(x)$ is the difference $f(x) - f_a(x)$ as a sum of W_0 wavelets. The two expansions are shown in Figures 6(e) and (f).



(a) (b)

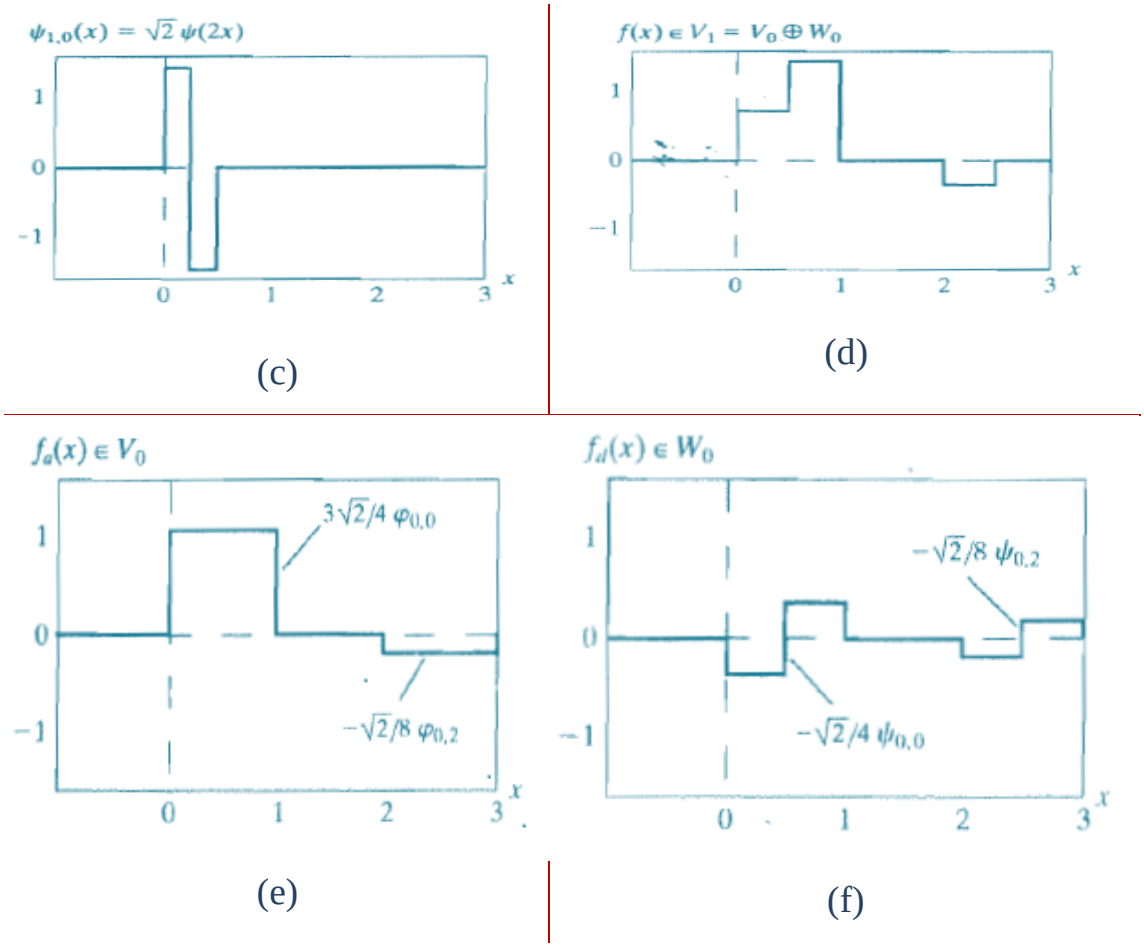


Figure 6: Haar Wavelet Functions in W_0 and W_1.

The wavelet transform has two types of transforms that are explained in the following section.

Wavelet Transform Types **:**

► The Continuous Wavelet Transform (CWT):

The Continuous wavelet transform (CWT) of a function *f(x)* includes a mother wavelet $\psi(x)$. The mother wavelet can be any continuous function, real or complex, that satisfies the following properties:

1. The total area under the curve of the function is zero, such that :

2. The total area of $|\psi(x)|^2$ is finite , such that

The CWT in 1-D of a square integrable function *f*(x) with respect to the wavelet $\psi(x)$ is a function $W_\psi(s,\tau)$ of two variables s,τ , and defined as:

W_ψ

where

$\psi_{s,\tau}(x$

$W_\psi(s,\tau)$: is the wavelet coefficient of the function *f(x)*.

s : is the scale parameter.

τ : is the position parameter.

The quantity $\frac{1}{\sqrt{s}}$ is a normalizing factor that guarantees the energy of $\psi(x)$ remains independent of *s* and τ, such that:

$$\int_{-\infty}^{\infty} \left|\psi_{s,\tau}(x)\right|^2 dx = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

For any *s*, $\psi_{s,\tau}(x)$ is a copy of $\psi_{s,0}(x)$ shifted τ along the time axis. Setting $\tau = 0$, τ is said to be ***translation*** parameter, such that :

$$\psi_{s,0}(x) = \frac{1}{\sqrt{s}}\psi\left(\frac{x}{s}\right) \quad for\ any\ s \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

The parameter *s* is said to be a ***scaling*** (dilation) parameter.

If $s > 1$ stretch the wavelet, while $0 < s < 1$ shrink the wavelet, as shown in **Figure 7.**

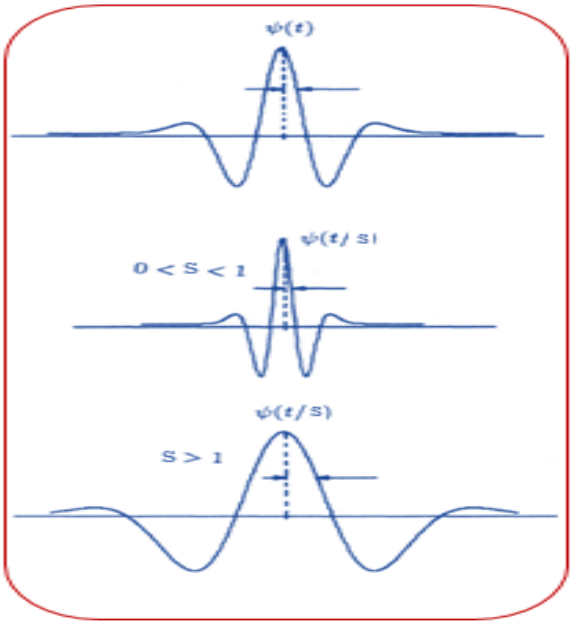


Figure 7: Typical Wavelet Family in Time and Frequency Domains

A function *f(x)* can be restored using the inverse continuous wavelet transformation by the formula:

$$f(x) = \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^{\infty} \frac{1}{s^2} W_\psi(s,\tau)\psi_{s,\tau}(x)d\tau\, ds \dots\dots\dots\dots (33)$$

where

$W_\psi(s,\tau)$: is a given wavelet coefficient.

$\psi_{s,\tau}(x)$: is Wavelet function.

C_ψ : is defined as

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega \dots\dots\dots\dots\dots\dots\dots (34)$$

and $\Psi(\omega)$ is the Fourier transform of $\psi(x)$, such that:

$$\Psi(\omega) = \int_{-\infty}^{\infty} \psi(x)e^{-i\omega x}dx \dots\dots\dots.. (35)$$

The inverse CWT exists if C_ψ is positive and finite. Since C_ψ is defined by means of Ψ, which itself is defined by means of the wavelet $\psi(x)$, the requirement that C_ψ be positive and finite imposes another restriction, called the ***admissibility condition***, on the choice of wavelet. So the wavelet is called admissible if $C_\psi < \infty$.

In applications such as **image compression**, the sampled data are discrete in time. Thus, a discrete representation of time and frequency is needed, which is called the ***Discrete Wavelet Transform*** (DWT). Before giving the definition of DWT, we need to explain the concept of ***wavelet series expansion***.

■ Wavelet Series Expansion

For a specific value $s = s_0$, by means of these subspaces, the Eq. (20) can be decomposed as

$$L^2(\mathbb{R}) = V_\infty = V_{s_0} \oplus W_{s_0} \oplus W_{s_0+1} \oplus W_{s_0+2} \oplus \dots \qquad (36)$$

This is shown in Figure 8.

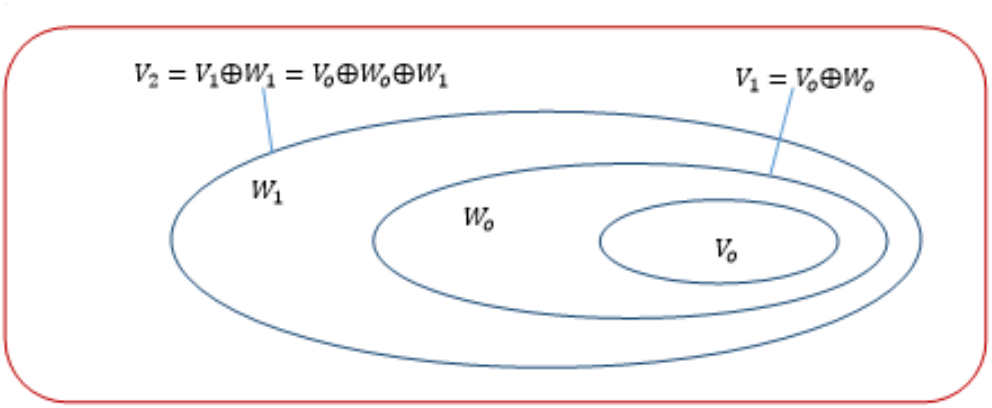


Figure 8: The Relationship Between Scaling and Wavelet Function Space

It indicates that any square-integrable function $f(x) \in L^2(\mathbb{R})$ can be expanded as a linear combination of both the scaling basis functions $\emptyset_{s_0,\tau}(x)$ and the wavelet basis functions $\psi_{s,\tau}(x)$, $s = s_0, s_0 + 1, \dots$. That is, each function $f(x) \in L^2(\mathbb{R})$ can be represented in its wavelet series expansion:

$$f(x) = \sum_{\tau} c_{s_0,\tau} \emptyset_{s_0,\tau}(x) + \sum_{s=s0}^{\infty} \sum_{\tau} d_{s,\tau}\, \psi_{s,\tau}(x) \dots\dots\dots\dots\dots\dots (37)$$

where $c_{s_0,\tau}$ is called ***approximation coefficient*** or ***scaling coefficients*** defined as:

$$c_{s_0,\tau}(\tau) = \langle f(x), \emptyset_{s_0,\tau}(x) \rangle = \int f(x)\, \emptyset_{s_0,\tau}(x)\, dx \dots\dots\dots\dots\dots (38)$$

and $d_{s,\tau}$ is called ***detail coefficient*** or ***wavelet coefficients*** defined as:

$$d_{s,\tau}(\tau) = \langle f(x), \psi_{s,\tau}(x) \rangle = \int f(x)\, \psi_{s,\tau}(x)\, dx \ \ldots\ldots\ldots \ (39)$$

The first term contained in the wavelet expansion of the function $f(x)$ Eq. (37) represents the ***approximation*** of the function at scale level s_0 by the linear combination of the scaling functions $\emptyset_{s_0,\tau}(x)$, and the summation with index *s* in the second term in the expansion is for the ***details*** of different levels contained in the function $f(x)$ approximated by the linear combination of the wavelet functions of progressively higher scales $s_0 + 1, s_0 + 2, \ldots$.

Example 3: Consider a continuous function $f(x)$, which is defined over the period $0 \le x \le 1$:

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

A shown in **Figure 9**. Using Harr wavelet and a starting scale $s_0 = 0$, each individual space $V_0, W_0, W_1, \ldots$ is spanned by different number of basis functions. Eqs. (38-39) can be used to compute the expansion coefficients:

Since $c_{s_0}(\tau) = \int f(x), \emptyset_{s_0,\tau}(x)\, dx$, then;

$$c_0(0) = \int_0^1 x^2\, \emptyset_{0,0}(x) dx = \int_0^1 x^2\, dx = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2\, \psi_{0,0}(x) dx = \int_0^{\frac{1}{2}} x^2\, dx \ - \int_{\frac{1}{2}}^1 x^2\, dx \ = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2\, \psi_{1,0}(x) dx = \int_0^{\frac{1}{4}} x^2\, \sqrt{2}\, dx \ - \int_{\frac{1}{4}}^{\frac{1}{2}} x^2\, \sqrt{2}\, dx \ = -\frac{\sqrt{2}}{32}$$

$$d_1(1) = \int_0^1 x^2\, \psi_{1,1}(x) dx = \int_0^{\frac{3}{4}} x^2\, \sqrt{2}\, dx \ - \int_{\frac{3}{4}}^1 x^2\, \sqrt{2}\, dx \ = -\frac{3\sqrt{2}}{32}$$

Substituting these values into Eq. (37), we get the wavelet series expansion

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\emptyset_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1=V_0\oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - -\frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2=V_1\oplus W_1=V_0\oplus W_0\oplus W_1} + \cdots$$

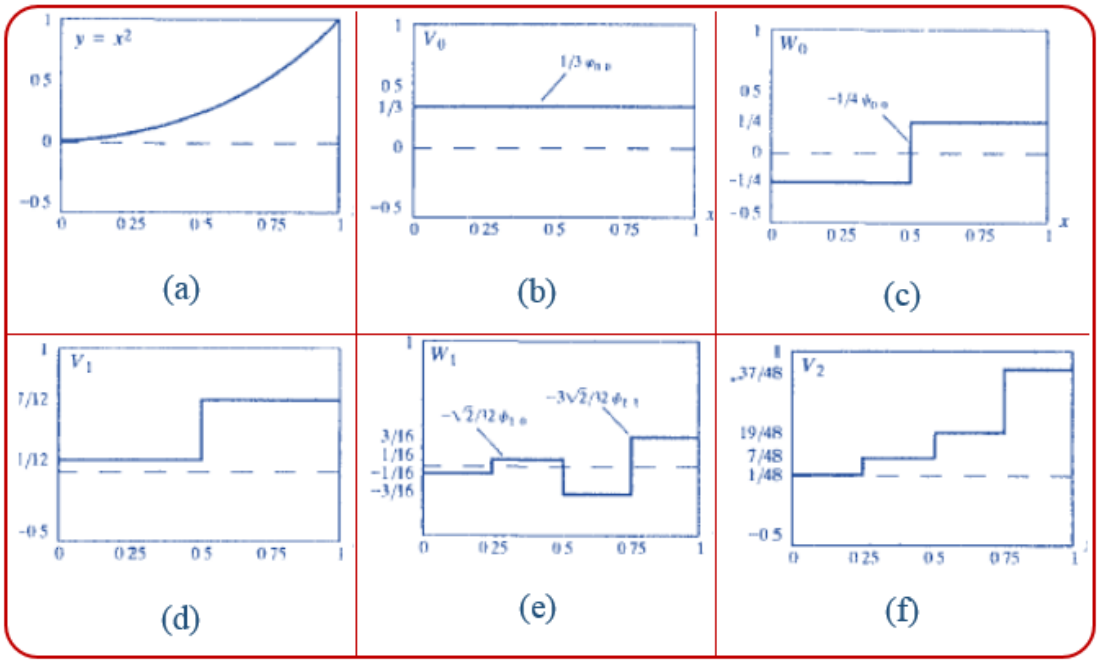


Figure 9: Wavelet Series Expansion of $f(x)=x^2$ Using Haar Wavelets

The first term in this expansion uses $c_0(0)$ to generate a subspace V_0 approximation of the function being expanded. This approximation is shown in Figure 9(b) and is the average value of the original function. The second term uses $d_0(0)$ to refine the approximation by adding a level of detail from subspace W_0. The added detail and resulting V_1 approximation are shown in Figures 9(c) and (d), respectively. Another level of detail is added by the subspace W_1 cofficients $d_1(0)$ and $d_1(1)$. This additional detail is shown in Figure 9(e) and the resulting V_2 approximation is depicted in Figure 9(f).

► The Discrete Wavelet Transform (DWT):

The DWT is an efficient algorithm for calculating the coefficients of the wavelet transform of a discrete series. It is like the Fourier series expansion, the wavelet series expansion of the previous section maps a function of a continuous variable into a sequence of

coefficients. If the function being expanded is discrete, the resulting coefficients are called the DWT.

The DWT in one – dimension (1-D) is given by:

$$W_{\emptyset}(s_0,\tau) = \frac{1}{\sqrt{M}}\sum_{x} f(x)\emptyset_{s_0,\tau}(x) \dots\dots\dots\dots\dots\dots\dots (40)$$

called the ***approximation*** or scaling coefficients, and

$$W_{\psi}(s,\tau) = \frac{1}{\sqrt{M}}\sum_{x} f(x)\psi_{s,\tau}(x) \ , For\ s \geq s_0 \dots\dots (41)$$

called the ***detail*** or wavelet coefficients.

The inverse of DWT in (1-D) is given by:

$$f(x) = \frac{1}{\sqrt{M}}\sum_{\tau} W_{\emptyset}(s_0,\tau)\emptyset_{s_0,\tau}(x) + \frac{1}{\sqrt{M}}\sum_{s=s_0}^{\infty}\sum_{\tau} W_{\psi}(s,\tau)\psi_{s,\tau}(x) \dots\dots (42)$$

Here $f(x)$, $\emptyset_{s_0,\tau}(x)$, and $\psi_{s,\tau}(x)$ are functions of discrete variable *x*, for example $f(x) = f(x_0 + k\Delta x)$ and $k = 0,1,2,\dots,M-1$. The factor $\frac{1}{\sqrt{M}}$ is normalizing factor.
Normally, we let $s_0 = 0$ and select *M* to be a power of 2 (i.e. $M = 2^s$) so that the summations are performed over $k = 0,1,2,\dots,M-1; s = 0,1,2,\dots,s-1$ and $\tau = 0,1,2,\dots,2^s - 1$.
The transform itself is composed of *M* coefficients, the minimum scale is *0* and the maximum scale is (s – 1).
The $W_{\emptyset}(s_0,\tau)$ and $W_{\psi}(s,\tau)$ of Eqs.(40) and (41) correspond to the $c_{s_0}(\tau)$'s and $d_s(\tau)$'s of the ***wavelet series expansion*** that explained in the previous section, that is Eqs. (38) and (39).

Example 4: To compute the 1-D DWT coefficients, consider the discrete function of four points: *f*(0) = 1, *f*(1) = 4, *f*(2) = -3 and *f*(3) = 0. *M* = 4, *s* = 2 and $s_0 = 0$. The summations are performed over *x* = 0,1,2,3 ; *s* = 0,1 and τ = 0 for *s* = 0 or τ =0,1 for *s* = 1. We will use the Haar scaling and wavelet functions and assume that the four sampling of $f(x)$are distributed over the support of the basis functions, which is 1. Substituting the four samples into Eqs. (40) and (41), we find :

$$W_{\emptyset}(0,0) = \frac{1}{2}\sum_{x=0}^{3} f(x)\emptyset_{0,0}(x)$$

$= \frac{1}{2}[1.1 + 4.1 - 3.1 + 0.1] = 1$ (since $\emptyset_{0,0}(x) = 1$ for $x = 0,1,2,3$)

$$W_{\psi}(s,\tau) = \frac{1}{\sqrt{M}}\sum_{x} f(x)\psi_{s,\tau}(x)$$

$$W_{\psi}(0,0) = \frac{1}{2}[1.1 + 4.1 - 3.(-1) + 0.(-1)] = 4$$

$$W_{\psi}(1,0) = \frac{1}{2}\left[1.\sqrt{2} + 4.(-\sqrt{2}) - 3.0 + 0.0\right] = -1.5\sqrt{2}$$

$$W_{\psi}(1,1) = \frac{1}{2}\left[1.0 + 4.0 - 3.\sqrt{2} + 0.(-\sqrt{2})\right] = -1.5\sqrt{2}$$

Thus, the DWT of the given four sample function relative to the Haar wavelet and scaling function is $[1,4, -1.5\sqrt{2}, -1.5\sqrt{2}]$, where the transform coefficients have been arranged in the order in which they were calculated.

To reconstruct the original function from its transform, using the Eq.(42):

$$f(x) = \frac{1}{\sqrt{M}}\sum_{\tau} W_{\emptyset}(s_0,\tau)\emptyset_{s_0,\tau}(x) + \frac{1}{\sqrt{M}}\sum_{s=s_0}^{\infty}\sum_{\tau} W_{\psi}(s,\tau)\psi_{s,\tau}(x)$$

Then;

$$f(x) = \frac{1}{2}\left[W_{\emptyset}(0,0)\emptyset_{0,0}(x) + W_{\psi}(0,0)\psi_{0,0}(x) + W_{\psi}(1,0)\psi_{1,0}(x) + W_{\psi}(1,1)\psi_{1,1}(x)\right]$$

For x = 0,1,2,3. If x = 0,

$$f(0) = \frac{1}{2}\left[1.1 + 4.1 - \left(1.5\sqrt{2}\right).\left(\sqrt{2}\right) - \left(1.5\sqrt{2}\right).0\right] = 1\ .$$

The one dimensional transforms (1-D) are extended easily to two dimensional transform (2-D) by using a 2-D scaling function $\phi(x,y)$, and three 2-D wavelets, $\psi^H(x,y), \psi^V(x,y), and\ \psi^D(x,y)$, so to reconstruct the signal $f(x,y)$ of size $M \times N$ in form the 2-D discrete wavelet coefficient$\left(W_f(s,m,n)\right)$, we found 2-D DWT pair becomes as:

$$W_{\emptyset}(s_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \emptyset_{s_0,m,n}(x, y) \qquad (43)$$

$$W_{\psi}^{i}(s, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi_{s,m,n}^{i}(x, y) \qquad (44)$$

where,

$$\emptyset_{s_0,m,n}(x, y) = 2^{\frac{s_0}{2}} \phi(2^{s_0} x - m, 2^{s_0} y - n) \qquad (45)$$

$$\psi_{s,m,n}^{i}(x, y) = 2^{\frac{s}{2}} \psi^{i}(2^{s} x - m, 2^{s} y - n) \qquad (46)$$

For $s > s_0$ and $i = \{H, V, D\}$.

As in 1-D case, s_0 is an arbitrary starting scale and the $W_{\emptyset}(s_0, m, n)$ coefficients define an approximation of $f(x, y)$ at scale s_0. The $W_{\psi}^{i}(s, m, n)$ coefficients add horizontal, vertical, and diagonal details for scales; $s > s_0$.

Usually, let $s_0 = 0$ and $N = M = 2^s$ so that $s = 0,1,2, \dots, s-1$ and $M = N = 0,1,2, \dots, 2^s - 1$. The inverse discrete wavelet ransform IDWT for 2-D is given by:

$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_{m} \sum_{n} W_{\phi}(s_0, m, n) \phi_{s_0,m,n}(x, y)$$

$$+ \frac{1}{\sqrt{MN}} \sum_{i-H,V,D} \sum_{s=s_0}^{\infty} \sum_{m} \sum_{n} W_{\psi}^{i}(s, m, n) \psi_{s,m,n}^{i}(x, y) \qquad (47)$$

Definition:

A two dimensional scaling function, $\phi(x, y)$, and three of two dimensional wavelet, $\psi^{H}(x, y), \psi^{V}(x, y), \psi^{D}(x, y)$, each is isolated by using the separable property to simplify the complexity computations. Where a two dimensional scaling function products of one dimensional scaling function as :

$$\phi(x, y) = \phi(x)\, \phi(y) \qquad (48)$$

and three with two dimensional wavelet, $\psi^{H}(x, y), \psi^{V}(x, y), \psi^{D}(x, y)$, each is the products of one dimensional wavelet function with respect to directionally sensitive wavelet as :

$$\psi^{H}(x, y) = \psi(x)\, \phi(y) \qquad (49)$$

$$\psi^V(x,y) = \phi(x)\,\psi(y) \qquad (50)$$

$$\psi^D(x,y) = \psi(x)\,\psi(y) \qquad (51)$$

These wavelet measure practical variations intensity or gray level variations for images along different directions as :

ψ^H : measures variations along columns (horizontal edges).

ψ^V : measures variations along rows (vertical edges).

ψ^D : measures variations along diagonals.

Remark:

The component that defined in Eq.(43) is called the ***approximation*** or the low pass component $[W_\phi]$ and characterizes the image's low frequency information; the component that defined in Eq.(44) is called the ***detail*** or high pass component $[W_\psi^H, W_\psi^V, W_\psi^D]$ and contains its high frequency information with various orientation.

■ *How the Wavelets Transform Works*

In order to explain how the wavelets transform works, we take the Haar wavelet transform, which is one of the simplest and basic transformations from the space domain to a local frequency domain. A Haar transform decomposes each signal into two components, one is called ***average*** (approximation) and the other is known as ***difference*** (detail). The following steps illustrate how the Haar transform can be used to calculate of a matrix of $m \times n$ size.

Step 1: Find the average of each pair of elements.

Step 2: Find the difference between each average and the elements it was calculated from.

Step 3: Fill the first half of the matrix with averages.

Step 4: Fill the second half of the matrix with differences.

Step 5: Repeat the process on the first half of the matrix.

Step 6: If the dimension is odd number, we can add row(coulum) of zero elements.

In order to give an idea of its implementation, the procedure of its application may be explained with the help of a simple example as shown below.

Example 5: Consider the 8x8 matrix ,

$$M = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

We start with an arbitrary vector representing one row of an 8×8 matrix.

Step 1: Average

$$y = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{bmatrix}$$

$$\frac{64+2}{2} \quad \frac{3+61}{2} \quad \frac{60+6}{2} \quad \frac{7+57}{2} \Rightarrow \quad \mathrm{B} = \begin{bmatrix} 33 & 32 & 33 & 32 \end{bmatrix}$$

Where *B* is called ***approximation coefficients***, and the results become the first four entries of our modified string y_1 .

Differencing

$$64 - 33 \quad 3 - 32 \quad 60 - 33 \quad 7 - 32 \Rightarrow \quad \mathrm{C} = \begin{bmatrix} 31 & -29 & 27 & -25 \end{bmatrix}$$

Where C is called ***detail coefficients***, and the results become the last four entries of y_1 .

$$y_1 = \begin{bmatrix} B & C \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 33 & 32 & 33 & 32 & 31 & -29 & 27 & -25 \end{bmatrix}$$

Step 2: Average

$$y_1 = \begin{bmatrix} \mathbf{33} & \mathbf{32} & \mathbf{33} & \mathbf{32} & 31 & -29 & 27 & -25 \end{bmatrix}$$

$$\frac{33+32}{2} \quad \frac{33+32}{2} \quad \Rightarrow \quad \mathrm{B} = \begin{bmatrix} 32.5 & 32.5 \end{bmatrix}$$

Differencing

$$33 - 32.5 \quad 33 - 32.5 \Rightarrow \quad \mathrm{C} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} \mathbf{32.5} & \mathbf{32.5} & 0.5 & 0.5 & 31 & -29 & 27 & -25 \end{bmatrix}$$

Step 3: Average

$$y_2 = \begin{bmatrix} \mathbf{32.5} & \mathbf{32.5} & 0.5 & 0.5 & 31 & -29 & 27 & -25 \end{bmatrix}$$

$$\frac{32.5 + 32.5}{2} \quad \Rightarrow \quad \mathrm{B} = \begin{bmatrix} 32.5 \end{bmatrix}$$

Differencing

$$32.5 - 32.5 \quad \Rightarrow \quad \mathrm{C} = \begin{bmatrix} 0 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} \mathbf{32.5} & 0 & 0.5 & 0.5 & 31 & -29 & 27 & -25 \end{bmatrix}$$

The Haar wavelet does this transformation to each row of the image matrix, and then again to every column in the matrix. The resulting matrix is known as the **Haar wavelet transform of the original matrix**. It is important to note at this point that this process is entirely reversible. It is this fact that makes it possible to retrieve the original matrix from the Haar wavelet transform of the matrix.

Apply average and differencing to the entire matrix *M*

First: the rows of matrix *M* is:

$$M = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 6 & 63 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 32.5 & 0 & .5 & .5 & 31 & -29 & 27 & -25 \\ 32.5 & 0 & -.5 & -.5 & -23 & 21 & -19 & 17 \\ 32.5 & 0 & -.5 & -.5 & -15 & 13 & -11 & 9 \\ 32.5 & 0 & .5 & .5 & 7 & -5 & 3 & -1 \\ 32.5 & 0 & .5 & .5 & -1 & 3 & -5 & 7 \\ 32.5 & 0 & -.5 & -.5 & 9 & -11 & 13 & -15 \\ 32.5 & 0 & -.5 & -.5 & 17 & -19 & 21 & -23 \\ 32.5 & 7 & .5 & -13.5 & -25 & 27 & -1 & 31 \end{bmatrix}$$

Second: the columns of matrix *M* is:

$$\begin{bmatrix} 32.5 & 0 & .5 & .5 & 31 & -29 & 27 & -25 \\ 32.5 & 0 & -.5 & -.5 & -23 & 21 & -19 & 17 \\ 32.5 & 0 & -.5 & -.5 & -15 & 13 & -11 & 9 \\ 32.5 & 0 & .5 & .5 & 7 & -5 & 3 & -1 \\ 32.5 & 0 & .5 & .5 & -1 & 3 & -5 & 7 \\ 32.5 & 0 & -.5 & -.5 & 9 & -11 & 13 & -15 \\ 32.5 & 0 & -.5 & -.5 & 17 & -19 & 21 & -23 \\ 32.5 & 7 & .5 & -13.5 & -25 & 27 & -1 & 31 \end{bmatrix} \Rightarrow \begin{bmatrix} 32.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & .5 & .5 & 27 & -25 & 23 & -21 \\ 0 & 0 & -.5 & -.5 & -11 & 9 & -7 & 5 \\ 0 & 0 & .5 & .5 & -5 & 7 & -9 & 11 \\ 0 & 0 & -.5 & -.5 & 21 & -23 & 25 & -27 \end{bmatrix} = N$$

A matrix N can be represented as a more concise manner, with one overall average in the upper left-hand corner of the matrix is called ***approximation coefficients***. The remaining components are all ***detail coefficients*** that now represent the amount of detail in that area of the matrix. Because we know this, we can eliminate some information from the given matrix and still be capable of attaining a fairly good approximation of the original matrix. Doing this it can choose some number δ and set equal to zero all elements with a magnitude less than δ.

Choosing $\delta = 5$, then eighteen of the detail coefficients (bold) are eliminated.

$$\begin{bmatrix} 32.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 27 & -25 & 23 & -21 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & -11 & 9 & -7 & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 7 & -9 & 11 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & 21 & -23 & 25 & -27 \end{bmatrix}$$

Remark 5:

The averaging and differencing method that we just discussed is very effective. Using linear algebra, we can use three matrices (A_1, A_2, A_3) that perform each of the three steps of the averaging and differencing process. In our previous Example (5), the transformation of y to y_1 can describe as

$y_1 = yA_1$

$$= [64 \;\; 2 \;\; 3 \;\; 61 \;\; 60 \;\; 6 \;\; 7 \;\; 57] \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$= [\,33 \;\; 32 \;\; 33 \;\; 32 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

It can next be shown that the transformation from y_1 to y_2 can be written as

$$y_2 = y_1 A_2$$

$$= [\,33 \;\; 32 \;\; 33 \;\; 32 \;\; 31 \;\; -29 \;\; 27 \;\; -25] \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\,32.5 \;\; 32.5 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

and lastly we can show that $y_3 = y_2 A_3$

$$= [\,32.5 \;\; 32.5 \;\; 0.5 \;\; 0.5 \;\; 31 - 29 \;\; 27 - 25] \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\,32.5 \;\; 0 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

This whole transformation can be done in one step by multiplying these three matrices together to obtain a single transform matrix $= A_1 A_2 A_3$. We can now multiply the original string by just one transform matrix to go directly from the original string to the final results of step 3.where W is,

$$W = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \end{bmatrix} \begin{bmatrix} 1/8 & 1/8 & 1/4 & 0 & 1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/4 & 0 & -1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & 1/2 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & -1/2 & 0 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & 1/2 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & -1/2 & 0 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & 1/2 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

$$= [\, 32.5 \;\; 0 \;\; 0.5 \;\; 0.5 \;\; 31 \;\; -29 \;\; 27 \;\; -25]$$

The following equations simplify this process of matrix multiplication of the averaging and differencing.

$$T = ((MW)^T W)^T$$

$$T = (W^T M^T W)^T$$

$$T = W^T (M^T)^T (W^T)^T$$

$$T = W^T M W$$

It is also important to note that since every column of the individual matrices that comprise W is orthogonal to every other column, the matrices are invertible. Thus,

$W^{-1} = A_3^{-1} A_2^{-1} A_1^{-1}$, then

$$N = W^T M W$$

$$M = (W^T)^{-1} N W^{-1}$$

where, M is original matrix

W is transforming matrix

N is compressed matrix.